Maximal Ellipsoid Method for Guaranteed Reachability of Unknown Fully Actuated Systems

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Abstract—**In the face of an adverse event, autonomous systems may undergo abrupt changes in their dynamics. In such an event, systems should be able to determine their continuing capabilities to then execute a provably completable task. This paper focuses on the scenario of a change in the system dynamics following an adverse event, aiming to determine the system's guaranteed performance capabilities by finding a set of states that are provably reachable by the system. While it is obviously impossible to exactly determine the reachable set without full knowledge of the system dynamics, we present a method of determining its under-approximation while assuming only partial knowledge of the system structure. Our technical approach relies on showing that an intersection of infinitely many ellipsoids — available velocity sets for each system consistent with the partial knowledge of the dynamics — is the same as an intersection of some finite collection of ellipsoids. This result enables us to find a maximal ellipsoid lying in such an intersection, yielding a set of velocities that the system is provably able to pursue regardless of its exact dynamics.**

I. INTRODUCTION

In the event where a physical system experiences significant damage, it is possible that there will consequently be an abrupt change in its dynamics. In such scenarios, understanding the continuing capabilities of the system is crucial, e.g., to produce emergency maneuvers. Consequently, this paper is concerned with developing the ability for a system to autonomously determine the set of *guaranteed* reachable states without a reliable dynamic model. We thus aim to find an under-approximation of the reachable set of states to determine what a system's capabilities are despite significant uncertainties in the model. The system can then assign provably completable tasks, thus ensuring continued survival and completion of the system's long-term mission. Following [1], we denote such a set as the *guaranteed reachable set* (GRS).

Prior initial work [1] focused on finding an under-approximation of the GRS for systems with unknown dynamics assuming knowledge of the local dynamics at a single point and Lipschitz bounds on the rate of change of said dynamics. However, these assumptions may often be limiting as they do not incorporate any additional physical and design knowledge that may be available about the system. In this paper, we make a first step at exploiting such knowledge, using information on the effect the system's actuators have on each state [2].

Following [1], our approach relies on the interpretation of a control system as a differential inclusion [3] whose right hand-side equals the set of velocities that the system can achieve at every state in the state space. The set of velocities that are provably reachable is the *guaranteed velocity set* (GVS). The GVS is, in an often-considered case where the system inputs lie in a unit ball, an intersection of infinitely many ellipsoids. Given its complex structure, previous work [1] under-approximated it. In this work, we show that the GVS can be represented as an intersection of only finitely many ellipsoids. Moreover, based on this fact, we also introduce a method that finds the optimal ellipsoidal under-approximation of the GVS by solving a semi-definite program with finitely many constraints. The simple geometric structure of our optimal ellipsoidal inclusion is then represented as a *known* control system whose reachable set produces a set of states guaranteed to be reachable by the original partially unknown system. We finish our work by showing the under-approximation is meaningful in practice by illustrating the results on a numerical example.

A. Prior Work

The work of this paper comes from a similar motivation, but significantly differs from that of robust and adaptive control [4], [5] and abstraction-based methods [6]. These methods, although central for many applications, largely attempt to ensure that a system will reach its original objective after a change in dynamics; they offer no guarantees for such reachability once the system's capabilities have changed significantly enough that the original objective is no longer reachable. Instead, our theory develops methods of determining states that are *provably* reachable.

Apart from [1] and its precursor paper [7] which were discussed above, existing methods that calculate the forward reachable set for systems with partially known dynamics assume that dynamics are generated by a finite number of uncertain parameters [8], [9] or undergo small disturbances [10]. These methods cannot calculate the reachable set of a nonlinear system whose dynamics are largely unknown, notwithstanding some knowledge of system design and physical laws. In contrast, our method can under-approximate the reachable set despite abrupt changes in the dynamics. Further similarities to our paper are found in techniques such as [11], [12], but those over-approximate the set of reachable states. Finally, classical data-driven learning methods [13], [14] collect data through repeated system runs, which cannot be executed onboard in short time intervals. Conversely, the proposed method directly exploits prior knowledge of physics and design and does not require multiple system runs, allowing for the potential of real-time computation.

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B. Notation

We denote the set of all $n \times m$ real matrices by $\mathbb{R}^{n\times m}$. For any vector v, $\|v\|$ denotes its Euclidean norm. For any matrix M, M^T denotes its transpose, $\|M\|$ denotes its Euclidean norm: $\|M\| = \max_{\|v\|=1} \|Mv\|$, and $\|M\|_{\max}$ denotes its max norm: $\|M\|_{\max} = \max_{ij} |m_{ij}|$ where m_{ij} are elements of M. Equivalently, $\|M\| = \sigma_1(M)$ where $\sigma_i(M)$ represents the i-th largest singular value of M. Additionally, $M \geq 0$ denotes a positive semi-definite matrix, and $\mathcal{S}^n_{++}$ the space of all symmetric positive definite matrices in $\mathbb{R}^{n\times n}$. We let I_n denote the identity matrix of dimension n, and $0_{m\times n}$ denote a zero matrix of appropriate dimensions. Set $GL(n)$ denotes the general linear group in $\mathbb{R}^{n\times n}$, i.e., all invertible $n \times n$ matrices. Symbol $\odot$ denotes the Hadamard product of two matrices. Notation $\mathbb{B}^n(a; b)$ denotes a closed ball in $\mathbb{R}^n$ centered at $a \in \mathbb{R}^n$ with radius $b \geq 0$ under the Euclidean norm. We define $C_A(M) = \{M + E \mid |e_{ij}| \leq |a_{ij}| \ \forall\ i, j\}$ where e_{ij} and a_{ij} denote elements of E and A respectively, and $[k] = \{1, \ldots, k\}$ for any $k \in \mathbb{Z}$. Notation $a + B\mathcal{X}$ where $a \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times m}$, and $\mathcal{X} \subseteq \mathbb{R}^m$ denotes the set $a + B\mathcal{X} = \{a + Bx \mid x \in \mathcal{X}\}$.

II. Problem Statement

Throughout the paper, we attempt to meaningfully under-approximate the reachable set of a nonlinear control-affine system $\mathcal{M}(f, G)$ defined by

$$\dot{x}(t) = f(x) + G(x(t))u(t), \quad x(0) = x_0, \tag{1}$$

where all $t \geq 0$, $x(t) \in \mathbb{R}^n$, functions $f : \mathbb{R}^n \to \mathbb{R}^n$, and $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are smooth enough to ensure the existence and uniqueness of a solution to (1) for the considered u, and admissible inputs $u(t) \in \mathcal{U} = \mathbb{B}^m(0; 1)$. Taking $\mathcal{U} = \mathbb{B}^m(0; 1)$, i.e., assuming that actuators can jointly generate inputs up to some maximal magnitude, is a common setting in constrained control [15], [16]. Without loss of generality, we assume $x_0 = 0$. The system is additionally known to satisfy the following assumption.

Assumption 1. The system $\mathcal{M}(f, G)$ is fully actuated at $x_0 = 0$, i.e., $m = n$ and $G(0) \in GL(n)$.

Full actuation is also assumed in previous work [7]; generalizing to the under-actuated case presents issues with determining the rank of $G(x)$ for $x \neq 0$ and is left for future work. We proceed by defining

$$\mathcal{A}(B) = \{A \mid a_{ij} \geq 0 \ \forall\ i, j, \\ \|A\| < \min_{\substack{D+B\in GL(n)\\ |d_{ij}|\leq a_{ij} \forall i,j}} \|(D + B)^{-1}\|^{-1}\}, \tag{2}$$

where d_{ij} and a_{ij} are elements of D, $A \in \mathbb{R}^{n\times n}$ respectively for all i, j. We can now define Assumption 2.

Assumption 2. Dynamics f are known, as well as $G(0)$ such that $G(0) \neq 0$. We assume that for all $x \in \mathbb{R}^n$, $G(x) \in C_{\Delta(x)}(G(0))$ where $\Delta(x)$ is known and $\Delta(x) \in \mathcal{A}(G(0))$.

As stated previously, we can compute $G(0)$ with an arbitrarily small error [17] and may know $\Delta(x)$ from known physical laws. In practice, it is possible that Assumption 2 is only known to hold on $\mathcal{D} \subseteq \mathbb{R}^n$. In such scenarios, we only consider the system's reachable set generated by trajectories that do not leave $\mathcal{D}$. Additionally, there is a simple sufficient condition to verify (2): by Weyl's inequality for matrices [18], the fact that $\|(B + D)^{-1}\|^{-1} = \sigma_n(B + D)$, and since $\|G(x) - G(0)\| < \|\Delta(x)\|$ by [19], if $\|\Delta(x)\| \leq \sigma_n(G(0))/2$, then $\Delta(x) \in \mathcal{A}(G(0))$.

We want to determine conditions under which $\text{rank}(G(0)) = \text{rank}(G(x))$ for some $\Delta(x)$. For the sake of completeness, we repeat and modify Lemma 1 in [1] in the following proposition which determines easily computable bounds that guarantee $\text{rank}(G(0)) = \text{rank}(G(x))$. Throughout the paper, $G^{-1}(0)$ denotes the inverse of $G(0)$.

Proposition 1. *If* $\|\Delta(x)\| < \|G^{-1}(0)\|^{-1}$, *then* $\text{rank}(G(x)) = \text{rank}(G(0))$.

Proof. Let A, B, D be as in equation (2) where $A = G(0)$, $B = \Delta(x)$, $D = G(x) - G(0)$. By Assumption 1 and equation (2), $\text{rank}(G(0)) = n$. By Weyl's inequality [18], $\sigma_i(G(x)) \geq \sigma_i(G(0)) - \|D\|$ for all $i \in [n]$. Thus, if $\|D\| < \sigma_n(G(0))$, then clearly $\sigma_i(G(x)) > 0$ for all $i \in [n]$, which implies $\text{rank}(G(x)) \geq n$, i.e., $\text{rank}(G(x)) = n$. Notice that $\sigma_n(G(0)) = \|G^{-1}(0)\|^{-1}$. By Assumption 2 and [19], $\|D\| \leq \|\Delta(x)\|$, so the claim holds. ■

We now focus on the problem of characterizing the unknown system's reachable set, as well as the set of velocities available to the system at every state.

A. Guaranteed Velocity Set

We define the *available velocity set* of the system $\mathcal{M}(f, G)$ at state x by $\mathcal{V}_x = f(x) + G(x)\mathcal{U}$, and introduce the following ODI:

$$\dot{x} \in \mathcal{V}_x = f(x) + G(x)\mathcal{U}, \quad x(0) = x_0. \tag{3}$$

As stated in [1], if a trajectory $\phi(\cdot; x_0)$ satisfies (3), then it obviously serves as a solution to the control system (1) for an admissible control input, and vice versa. We use the classical notion of a solution of ODE (1) and ODI (3) — as defined, e.g., in [20] — where the relevant equation or inclusion needs to hold for almost every t. Given Assumption 2, set $\mathcal{V}_{x_0} = \mathcal{V}_0$ is known. To account for velocities available in a system with unknown dynamics, we define the *guaranteed velocity set (GVS)* below:

$$\mathcal{V}^{\mathcal{G}}_x = f(x) + \bigcap_{\hat{G} \in C_{\Delta(x)}(G(0))} \hat{G}(x)\mathcal{U} \subseteq \mathcal{V}_x. \tag{4}$$

The GVS $\mathcal{V}^{\mathcal{G}}_x$ is the set of all velocities that can be taken by *all* systems consistent with the assumed knowledge of the dynamics. It is thus natural to consider the following ODI:

$$\dot{x} \in \mathcal{V}^{\mathcal{G}}_x, \quad x_0 = 0. \tag{5}$$

If $\mathcal{V}^{\mathcal{G}}_{\phi(T;x_0)} = \emptyset$ for some T, we will use the convention that the trajectory of (5) ceases to exist by time T. We now continue towards guaranteed reachability, and discuss how (5) can help under-approximate the system's reachable set.

B. Guaranteed Reachable Set

We want to under-approximate the system's set of reachable states while exploiting the knowledge of unforced dynamics f and bounds on $G(x)$ obtained from $C_{\Delta(x)}(G(0))$. Let $T \geq 0$. Let us define the *(forward) reachable set* $\mathcal{R}^{f,\hat{G}}(T, x_0) = \{\phi_u^{f,\hat{G}}(t; x_0) \mid u : [0,T] \to \mathcal{U}, t \in [0,T]\}$, where $\phi_u^{f,\hat{G}}(\cdot; x_0)$ denotes the controlled trajectory of the system $\mathcal{M}(f, \hat{G})$ with control signal u and $\phi_u^{f,\hat{G}}(0; x_0) = x_0$.

We describe the *guaranteed reachable set (GRS)* as

$$\mathcal{R}^{\mathcal{G}}(T, 0) = \bigcap_{\hat{G} \in C_{\Delta(x)}(G(0))} \mathcal{R}^{f,\hat{G}}(T, 0). \tag{6}$$

Equation (6) characterizes the set of all states that are reachable by any system consistent with our knowledge of the system dynamics. Our paper's central problem is given as follows:

Problem. Determine or meaningfully under-approximate the GRS.

To solve this problem, we turn back to the GVS defined above. The overall idea is to provide an under-approximation for set $\mathcal{V}_x$ using sets $C_{\Delta(x)}(G(0))$ and $\mathcal{V}_0$, then to use this under-approximation to arrive at a control system whose reachable set under-approximates the GRS. The following proposition, proved in [7], holds directly from (6) and (4).

Proposition 2. *Let $T \geq 0$. If a trajectory $\phi : [0, +\infty) \to \mathbb{R}^n$ satisfies* (5) *at all times $t \leq T$, then $\phi(T) \in \mathcal{R}^{\mathcal{G}}(T, 0)$.*

We denote the reachable set of (5) as $\mathcal{R}_{\mathcal{V}_x^{\mathcal{G}}}(T, 0)$. Note that by Proposition 2, clearly $\mathcal{R}_{\mathcal{V}_x^{\mathcal{G}}}(T, 0) \subseteq \mathcal{R}^{\mathcal{G}}(T, 0)$, however these sets are not necessarily equal [7]. Establishing conditions for the equality of the reachable set of (5) and $\mathcal{R}^{\mathcal{G}}(T, 0)$ is an open problem for future work. In the next section, we begin formulating the theory that can be used to calculate the reachable set of (5).

III. Finite Perturbation Theorem

The Finite Perturbation Theorem, which serves as the focal point of this paper, determines conditions under which calculating $\mathcal{V}_x^{\mathcal{G}}$ can be reduced to characterizing a set with finitely many constraints. We begin by describing the supporting lemmata needed to prove this theorem.

A. Supporting Lemmata

The first step is to derive conditions under which the sign of elements of $u \in \mathcal{U}$ are identical to satisfy conditions required for Lemma 2 to hold.

Lemma 1. *Let $A \in GL(n)$ and $E \in \mathbb{R}^{n \times n}$ so that*

$$E = \begin{bmatrix} 1 & 0_{1\times(n-1)} \\ & 0_{(n-1)\times n} \end{bmatrix}. \tag{7}$$

Let $0 \leq \delta < \|A^{-1}\|^{-1}$, $v \in \mathbb{R}^n$, and $u^\delta, u^{-\delta} \in \mathbb{B}^n(0;1)$ be such that

$$v = (A + \delta E)u^\delta, \quad v = (A - \delta E)u^{-\delta}. \tag{8}$$

If we denote $u^\delta := \begin{bmatrix} u_1^\delta \\ \hat{u}^\delta \end{bmatrix}$ and $u^{-\delta} := \begin{bmatrix} u_1^{-\delta} \\ \hat{u}^{-\delta} \end{bmatrix}$ with $u_1^\delta, u_1^{-\delta} \in \mathbb{R}$, assume $u_1^\delta \neq 0$ and $u_1^{-\delta} \neq 0$. Then $\mathrm{sign}(u_1^\delta) = \mathrm{sign}(u_1^{-\delta})$.

Proof. Given that $\|\delta E\| = \delta < \|A^{-1}\|^{-1}$, following the steps of the proof of Proposition 1, we observe that $\mathrm{rank}(A) = \mathrm{rank}(A + \delta E)$. Hence, $A + \delta E \in GL(n)$. We continue by expressing u^δ and $u^{-\delta}$ in the following terms:

$$u^\delta = (A + \delta E)^{-1}(A + \delta E)u^\delta = (A + \delta E)^{-1}v,$$
$$u^{-\delta} = (A - \delta E)^{-1}(A - \delta E)u^{-\delta} = (A - \delta E)^{-1}v.$$

It follows that

$$\begin{aligned} \begin{bmatrix} u_1^\delta \\ 0_{(n-1)\times 1} \end{bmatrix} &= E(A + \delta E)^{-1}v, \\ \begin{bmatrix} u_1^{-\delta} \\ 0_{(n-1)\times 1} \end{bmatrix} &= E(A - \delta E)^{-1}v. \end{aligned} \tag{9}$$

Clearly, $v \in \mathbb{R}^n = \mathrm{Im}(A + \delta E) = \mathrm{Im}(A)$, so we know there exists a vector u such that $v = Au$. Thus,

$$\begin{bmatrix} u_1 \\ 0_{(n-1)\times 1} \end{bmatrix} = EA^{-1}v. \tag{10}$$

If $\mathrm{sign}(u_1^\delta) = \mathrm{sign}(u_1)$ and $\mathrm{sign}(u_1^{-\delta}) = \mathrm{sign}(u_1)$, then $\mathrm{sign}(u_1^\delta) = \mathrm{sign}(u_1^{-\delta})$ must hold. It is trivial to show that if $|b| > |a - b|$ for any $a, b \in \mathbb{R}$, then $\mathrm{sign}(a) = \mathrm{sign}(b)$. Thus, if $|u_1^\delta| > |u_1 - u_1^\delta|$, i.e., by (9) and (10), $\|E(A+\delta E)^{-1}v\| > \|E(A^{-1} - (A + \delta E)^{-1})v\|$, then $\mathrm{sign}(u_1^\delta) = \mathrm{sign}(u_1)$.

For invertible matrices $M, N \in GL(n)$, the property $M^{-1} - N^{-1} = M^{-1}(N - M)N^{-1}$ holds. Taking $M = A$ and $N = (A+\delta E)$, we arrive at $\|E(A^{-1} - (A+\delta E)^{-1})v\| = \|EA^{-1}((A + \delta E) - A)(A + \delta E)^{-1}v\| = \delta\|EA^{-1}E(A + \delta E)^{-1}v\|$. Hence, if we prove $\delta\|EA^{-1}E(A + \delta E)^{-1}v\| < \|E(A + \delta E)^{-1}v\|$, then we have shown that $\mathrm{sign}(u_1^\delta) = \mathrm{sign}(u_1)$. Applying the product inequality for matrices, we see that since $\delta\|EA^{-1}E(A+\delta E)^{-1}v\| \leq \delta\|EA^{-1}\|\|E(A+\delta E)^{-1}v\|$ and $\|E(A + \delta E)^{-1}v\| = |u_1^\delta| \neq 0$, it suffices to show $\delta\|EA^{-1}\| < 1$.

Clearly $\|E\| = 1$ from (7), thus utilizing the product inequality and $\delta < \|A^{-1}\|^{-1}$ implies $\mathrm{sign}(u_1^\delta) = \mathrm{sign}(u_1)$. An analogous set of steps can prove that $\mathrm{sign}(u_1^{-\delta}) = \mathrm{sign}(u_1)$, which implies $\mathrm{sign}(u_1^\delta) = \mathrm{sign}(u_1^{-\delta})$. ■

We now use Lemma 1 as a stepping stone to describe an intersection of infinitely many ellipsoids $B_\alpha\mathcal{U}$ whose matrices B_α differ in a single element by an intersection of only two ellipsoids. Such a result is proved in the following lemma; in the context of guaranteed velocity sets, we later apply it to perturbations of matrix $G(0)$.

Lemma 2. *Let $A, E, \delta, u^\delta, u^{-\delta}$, and v be as in Lemma 1. Then, for every $\alpha \in [-1, 1]$, there exists $u^{\alpha\delta} \in \mathbb{B}^n(0;1)$ so that*

$$v = (A + \alpha\delta E)u^{\alpha\delta}. \tag{11}$$

Proof. Let us denote $v = \begin{bmatrix} v_1 \\ \hat{v} \end{bmatrix}$, with $v_1 \in \mathbb{R}$, and $A = \begin{bmatrix} a & \hat{A}_1 \\ & \hat{A} \end{bmatrix}$, where $a \in \mathbb{R}$, $\hat{A}_1 \in \mathbb{R}^{1\times(n-1)}$, $\hat{A} \in \mathbb{R}^{(n-1)\times n}$.

Let $\lambda \in [0,1]$. We set

$$u^{\alpha\delta} = \lambda u^{\delta} + (1-\lambda)\, u^{-\delta} = \begin{bmatrix} \lambda u_1^{\delta} + (1-\lambda)\, u_1^{-\delta} \\ \lambda \hat{u}^{\delta} + (1-\lambda)\, \hat{u}^{-\delta} \end{bmatrix},$$

which by convexity, guarantees $u^{\alpha\delta} \in \mathbb{B}^m(0;1)$. We will show that there exists λ such that (11) holds.

As $\hat{v} = \hat{A}u^{\delta} = \hat{A}u^{-\delta}$, $\hat{v}$ also equals $\hat{A}\left(\lambda u^{\delta} + (1-\lambda)u^{-\delta}\right) = \hat{A}u^{\alpha\delta}$ for any λ. It remains to prove that there exists λ such that

$$v_1 = (a+\alpha\delta)(\lambda u_1^{\delta} + (1-\lambda)u_1^{-\delta}) + \hat{A}_1(\lambda \hat{u}^{\delta} + (1-\lambda)\, \hat{u}^{\delta}). \tag{12}$$

From (8) we arrive at $\hat{A}_1 \hat{u}^{\delta} = v_1 - (a+\delta)u_1^{\delta}$ and $\hat{A}_1 \hat{u}^{-\delta} = v_1 - (a-\delta)u_1^{-\delta}$. Plugging these equalities into (12), we obtain that it is equivalent to

$$\delta\lambda(\alpha-1)u_1^{\delta} + \delta(1-\lambda)(\alpha+1)u_1^{-\delta} = 0. \tag{13}$$

If 0 is in the convex hull of $(\alpha-1)u_1^{\delta}$ and $(a+1)u_1^{-\delta}$, then there exists $\lambda \in [0,1]$ that satisfies (13). Given that $\alpha - 1 \le 0$ and $\alpha + 1 \ge 0$, from Lemma 1, $\mathrm{sign}((\alpha-1)u_1^{\delta}) = -\mathrm{sign}((\alpha+1)u_1^{-\delta})$ or $(\alpha-1)u_1^{\delta} = 0$ or $(\alpha+1)u_1^{-1} = 0$. Thus, 0 is indeed in such a convex hull, and (13) is satisfied. ∎

To simplify notation, Lemmas 1 and 2 speak of perturbations of the "top-left element" of A. However, the same results, with an analogous proof, also obviously hold for any other single element of A.

We now move to applying Lemmas 1 and 2 to matrices from the unknown function G.

B. Finite Perturbation Theorem

For Lemmas 1 and 2, we require the perturbation $\delta < \|A^{-1}\|^{-1}$. We begin with a remark that $\|\Delta\|_{\max} \le \|\Delta\|$ for all matrices Δ (see, e.g., [21]). Hence, if $\|\Delta\| < \|A^{-1}\|^{-1}$, all elements of $\Delta(x)$ are smaller than $\|A^{-1}\|^{-1}$.

The Finite Perturbation Theorem derives a method to represent an intersection of infinitely many ellipsoids by an intersection of only finitely many of them. Naturally, we will later apply it to simplify the computation of $\mathcal{V}_x^{\mathcal{G}}$ when $\Delta(x)$ is small enough.

Theorem 1 (Finite Perturbation Theorem)**.** *Let $u \in \mathcal{U} = \mathbb{B}^n(0;1)$, $A \in GL(n)$ and let $P_1, P_2, \ldots, P_{2^{n^2}}$ denote all matrices in $\mathbb{R}^{n\times n}$ with either 1 or -1 in every element. Let $\Delta \in \mathcal{A}(A)$ satisfy $\|\Delta\| < \|A^{-1}\|^{-1}$. We define $\mathcal{V} = \bigcap_{\hat{A}\in C_\Delta(A)} \hat{A}\mathcal{U}$. If we define $\Delta_k = \Delta \odot P_k$ for all $k \in [2^{n^2}]$, then*

$$\mathcal{V} = \bigcap_{k\in[2^{n^2}]} (A + \Delta_k)\mathcal{U}. \tag{14}$$

Proof. Let δ_{ij} denote elements of Δ. By the remark at the beginning of this section, δ_{ij} satisfies the conditions of Lemmas 1 and 2 for all i, j. Let $E_{ij} \in \mathbb{R}^{n\times n}$ be a matrix with all zeros, except 1 in element (i,j).

Consider any matrix $\hat{A} \in C_\Delta(A)$. Then, $\hat{A} = A + \sum_{i,j} \varepsilon_{ij}E_{ij}$, where $|\varepsilon_{ij}| \le \delta_{ij}$ for all i, j.

By first considering matrices $(A + \sum_{(i,j)\neq(1,1)} \varepsilon_{ij}E_{ij}) + \delta_{11}E_{11}$ and $(A + \sum_{(i,j)\neq(1,1)} \varepsilon_{ij}E_{ij}) - \delta_{11}E_{11}$ for Lemma 2, noting that $\delta_{11} < \|A + \sum_{(i,j)\neq(1,1)} \varepsilon_{ij}E_{ij}\|$, we obtain that $((A + \sum_{(i,j)\neq(1,1)} \varepsilon_{ij}E_{ij}) + \delta_{11}E_{11})\mathcal{U} \cap ((A + \sum_{(i,j)\neq(1,1)} \varepsilon_{ij}E_{ij}) + \delta_{11}E_{11})\mathcal{U} \subseteq (A + \sum_{i,j} \varepsilon_{ij}E_{ij})\mathcal{U}$ for all $\varepsilon \in [-\delta_{11}, \delta_{11}]$.

We now proceed onwards: each of the sets $((A + \sum_{(i,j)\neq(1,1)} \varepsilon_{ij}E_{ij}) \pm \delta_{11}E_{11})\mathcal{U}$ is, again by Lemma 2, a superset of $((A + \sum_{(i,j)\neq(1,1),(i,j)\neq(1,2)} \varepsilon_{ij}E_{ij}) \pm \delta_{11}E_{11} + \delta_{12}E_{12})\mathcal{U} \cap ((A + \sum_{(i,j)\neq(1,1),(i,j)\neq(1,2)} \varepsilon_{ij}E_{ij}) \pm \delta_{11}E_{11} - \delta_{12}E_{12})\mathcal{U}$. Continuing onwards, we finally obtain

$$\mathcal{V} \subseteq \bigcap_{k\in[2^{n^2}]} (A + \Delta_k)\mathcal{U} \subseteq \hat{A}\mathcal{U}$$

for every $\hat{A} \in C_\Delta(A)$, thus proving (14). ∎

By applying Theorem 1 to $A = G(0)$ and function Δ as defined in Assumption 2, the Finite Perturbation Theorem states that if $\|\Delta(x)\| < \|G^{-1}(0)\|^{-1}$, then

$$\mathcal{V}_x^{\mathcal{G}} = f(x) + \bigcap_{k\in[2^{n^2}]} (G(0) + \Delta_k(x))\mathcal{U}. \tag{15}$$

To provide an intuitive understanding of Theorem 1, we follow with an illustrative example.

Example. Let us consider matrix $A = \begin{bmatrix} 18 & 0 \\ -6 & 7 \end{bmatrix}$ and maximal perturbations $\Delta = \begin{bmatrix} 4.5 & 0 \\ 2.5 & 4.5 \end{bmatrix}$; notice that $\|\Delta\| < \|G^{-1}(0)\|^{-1}$. Consider $\bar{A} = \begin{bmatrix} 16 & 0 \\ -5 & 5.5 \end{bmatrix} \in C_\Delta(A)$. We will use Theorem 1 to compute $\cap_{\hat{A}\in C_\Delta(A)} \hat{A}\mathcal{U}$.

Let $u^{+--}, u^{---} \in \mathcal{U}$. Using the same notation as Lemma 2, let $\delta_{11} = 4.5$. By Lemma 2, if

$$v = \begin{bmatrix} 13.5 & 0 \\ -8.5 & 2.5 \end{bmatrix} u^{+--} = \begin{bmatrix} 22.5 & 0 \\ -8.5 & 2.5 \end{bmatrix} u^{---},$$

then there exists $u^{--} \in \mathcal{U}$ such that

$$v = \begin{bmatrix} 16 & 0 \\ -8.5 & 2.5 \end{bmatrix} u^{--}.$$

Analogously,

$$v = \begin{bmatrix} 13.5 & 0 \\ -3.5 & 2.5 \end{bmatrix} u^{++-} = \begin{bmatrix} 22.5 & 0 \\ -3.5 & 2.5 \end{bmatrix} u^{-+-}$$

implies that there exists $u^{+-} \in \mathcal{U}$ such that

$$v = \begin{bmatrix} 16 & 0 \\ -3.5 & 2.5 \end{bmatrix} u^{+-}.$$

Combining u^{+-} and u^{--} for Lemma 2, we obtain that there exists $u^{-} \in \mathcal{U}$ such that

$$v = \begin{bmatrix} 16 & 0 \\ -5 & 2.5 \end{bmatrix} u^{-}.$$

The above procedure is shown in Fig. 1; perturbations of element $(1,2)$ are skipped as $\delta_{12} = 0$.

An analogous procedure for vectors $u^{--+}, u^{+-+}, u^{-++}, u^{+++}$ would yield

$$v = \begin{bmatrix} 16 & 0 \\ -5 & 11.5 \end{bmatrix} u^{+}.$$

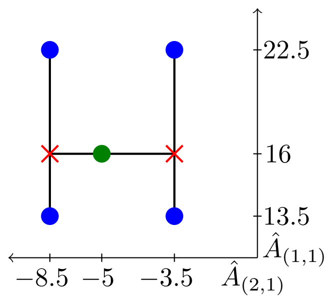

Fig. 1: The first two steps in the illustration of the Finite Perturbation Theorem. The elements of the initial matrices $A + \Delta_k$ are denoted in blue, the elements of the matrices obtained after the first step are denoted in red, and those obtained after two steps in green.

Finally, combining u^+ and u^- for Lemma 2, we obtain that there exists $u \in \mathcal{U}$ such that $v = \bar{A}u$. Fig. 2 illustrates set $\cap_{\hat{A} \in C_\Delta(A)} \hat{A}\mathcal{U}$ computed in two ways: with and without Theorem 1. The notation corresponds to that of the guaranteed velocity sets, with $G(0) = A$, $\Delta(x) = \Delta$, and $\mathcal{V} = \mathcal{V}_x^{\mathcal{G}}$.

The Finite Perturbation Theorem interprets $\mathcal{V}_x^{\mathcal{G}}$ as an intersection of just finitely many ellipsoids. However, its geometry is still nontrivial. Hence, we proceed to develop an optimal under-approximation of $\mathcal{V}_x^{\mathcal{G}}$ by exploiting this result.

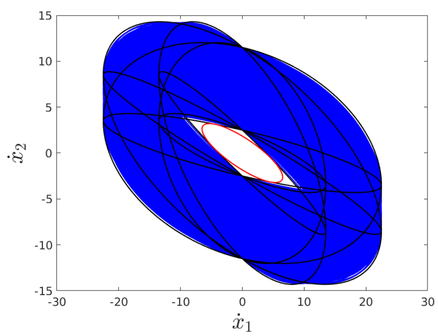

Fig. 2: Velocity sets in the example. The blue curves represent the boundaries of the ellipsoids $\hat{A}\mathcal{U}$, i.e., the available velocity sets for all systems such that $\hat{G}(x) \in C_{\Delta(x)}(G(0))$. The intersection of these sets produces $\mathcal{V}_x^{\mathcal{G}}$ (white). The boundary of $\mathcal{V}_x^{\mathcal{G}}$ is indeed formed by black curves, corresponding to 2^{n^2} ellipsoids from Theorem 1. The red curve is the boundary of the optimal ellipsoidal inner-approximation of $\mathcal{V}_x^{\mathcal{G}}$ introduced in Section IV.

IV. Maximal Ellipsoid

We can express $\mathcal{V}_x^{\mathcal{G}}$ as a finite intersection of ellipsoids satisfying the assumptions of Theorem 1. We draw on [22] by exploiting this new representation to solve an optimization problem that will determine the maximal ellipsoid contained in $\mathcal{V}_x^{\mathcal{G}}$. With this ellipsoid, we can derive a control system whose reachable set is contained in the GRS.

Theorem 2. *Let A and Δ_k be as in Theorem 1. Let $U\Sigma V^T$ be the singular value decomposition of A and let $\Sigma_k = U^T(A+\Delta_k)V$ and $A_k = (\Sigma_k^{-1})^T\Sigma_k^{-1}$. Let $\mathcal{E}_k = (A+\Delta_k)\mathcal{U}$. Then, an ellipsoid $\mathcal{E}$ of maximal volume such that $\mathcal{E} \subseteq \bigcap_k \mathcal{E}_k$ is given by $\mathcal{E} = UBV^T\mathcal{U}$ where B is the solution to*

$$\begin{aligned} &\underset{B \in \mathcal{S}_{++}^n, \lambda_1, \dots, \lambda_{2^{n^2}} \in \mathbb{R}}{\text{minimize}} \quad \log\det B^{-1} \\ &\text{subject to} \begin{bmatrix} -\lambda_k + 1 & 0 & 0 \\ 0 & \lambda_k I_n & B \\ 0 & B & A_k^{-1} \end{bmatrix} \geq 0 \\ &\text{for all } k \in [2^{n^2}]. \end{aligned} \tag{16}$$

Proof. By representing $\mathcal{E}_k$ via convex quadratic inequalities, the claim follows directly from Section 8.4.2 in [22]. ∎

The next result naturally follows from Theorem 2.

Corollary 1. *Let $x \in \mathbb{R}^n$. Let $B(x)$ be the solution of (16), with $A = G(0)$, $\Delta = \Delta(x)$. Let $U\Sigma V^T$ be the singular value decomposition of $G(0)$. Then, $UB(x)V^T\mathcal{U} \subseteq \mathcal{V}_x^{\mathcal{G}}$.*

We now use the result of Theorem 2 and Corollary 1 to produce a geometrically simple control system whose reachable set under-approximates the GRS.

Theorem 3. *For all $x \in \mathbb{R}^n$, let $B(x)$ be the solution of (16), with $A = G(0)$, $\Delta = \Delta(x)$. Let $U\Sigma V^T$ be the singular value decomposition of $G(0)$. Consider the control system*

$$\dot{x} = f(x) + UB(x)V^T u, \quad x(0) = x_0, \tag{17}$$

where $u \in \mathcal{U}$. If we define $\mathcal{R}_{opt}(T, x_0)$ as the reachable set of (17) at time T, then $\mathcal{R}_{opt}(T, x_0) \subseteq \mathcal{R}^{\mathcal{G}}(T, x_0)$.

Proof. Proposition 2 and Corollary 1 show that $\mathcal{R}_{opt}(T, x_0) \subseteq \mathcal{R}^{\mathcal{G}}(T, x_0)$. ∎

Theorem 3 provides a guaranteed under-approximation of the true reachable set of a partially known system; another is provided directly by the remark under Proposition 2. In the following section, we numerically solve for the under-approximated reachable sets to illustrate how the theory can be practically implemented, and introduce a new heuristic to reduce the computational load of these two methods.

V. Numerical Example

We consider a system with dynamics

$$\dot{x} = \begin{bmatrix} 18 - x_1 & 0 \\ -6 - x_2 & 7 - x_1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

These dynamics are partially unknown when calculating the reachable set, and all that is known is

$$G(0) = \begin{bmatrix} 18 & 0 \\ -6 & 7 \end{bmatrix}, \quad \Delta(x) = \begin{bmatrix} |x_1| & 0 \\ |x_2| & |x_1| \end{bmatrix}.$$

Fig. 3 shows three under-approximations of the GRS and the unknown system's true reachable set at $T = 0.2$.

The largest is $\mathcal{R}_{\mathcal{V}_x^{\mathcal{G}}}(0.2, 0)$ directly calculated as the reachable set of $\dot{x} \in \mathcal{V}_x^{\mathcal{G}}$, with $\mathcal{V}_x^{\mathcal{G}}$ calculated for every x using Theorem 1. The second largest set is $\mathcal{R}_{opt}(0.2, 0)$, i.e., the reachable set computed using Theorem 3.

Finally, we introduce a new reachable set, $\mathcal{R}_{con}(0.2, 0)$, obtained by taking $\Delta(4.5, 2.5)$, corresponding to the example in Section III, and plugging that Δ at every x to obtain a

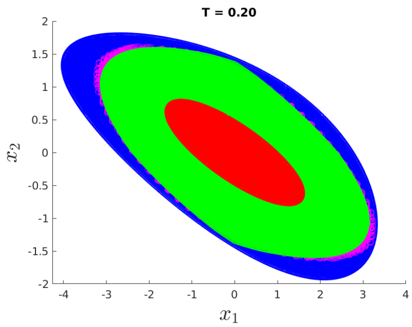

Fig. 3: True reachable set (blue) of the example system with under-approximations $\mathcal{R}_{\mathcal{V}_x^{\mathcal{G}}}(T,0)$ (purple), $\mathcal{R}_{opt}(T,0)$ (green) and $\mathcal{R}_{con}(T,0)$ (red) numerically computed for $T=0.2$ seconds.

state-invariant under-approximation of $\mathcal{V}_x^{\mathcal{G}}$ using Theorem 3. We choose such a Δ based on the additional knowledge that the system must satisfy $|x_1(T)| \leq 4.5$ and $|x_2(T)| \leq 2.5$; such knowledge might again be obtained from system design and physics, or computed using Lipschitz bounds to determine the over-approximated velocity set for every x, similarly to [1], [23].

All reachable sets are computed numerically through a Monte Carlo simulation using the ode45 function in MATLAB; throughout all the trajectories, $\Delta(x)$ indeed lies in $\mathcal{A}(G(0))$. We use such a rudimentary method to avoid technical issues of currently available solvers such as CORA [24], particularly for computing $\mathcal{R}_{\mathcal{V}_x^{\mathcal{G}}}(0.2,0)$ which utilizes a geometrically nontrivial shape of the velocity set that cannot easily be processed using such solvers.

While $\mathcal{R}_{con}(0.2,0)$, $\mathcal{R}_{opt}(0.2,0)$, and $\mathcal{R}_{\mathcal{V}_x^{\mathcal{G}}}(0.2,0)$ are increasingly better approximations of the system's true reachable set, the price paid is one of increased computational complexity. Namely, while the key to our work is our description of $\mathcal{V}_x^{\mathcal{G}}$ as an intersection of finitely many ellipses, the calculation of $\mathcal{R}_{\mathcal{V}_x^{\mathcal{G}}}(0.2,0)$ is still infeasible in real time because $\mathcal{V}_x^{\mathcal{G}}$ is difficult to describe in a simple form. Thus, $\mathcal{R}_{opt}(0.2,0)$ uses simple ellipsoids as velocity sets, but requires solving a different optimization problem at every state. While $\mathcal{R}_{opt}(0.2,0)$ can be computed with a lower runtime, the computation of either set takes hours. In contrast, calculating $\mathcal{R}_{con}(0.2,0)$ requires solving only one optimization problem in seconds, resulting in a smaller set that is feasible for real-time computation.

VI. Conclusion

This paper considers the problem of under-approximating the reachable set of a system with partially unknown dynamics. The theory centers around the Finite Perturbation Theorem, which takes advantage of knowledge of the effect of the system's actuators to represent the system's guaranteed available velocities as an intersection of finitely many ellipsoids. We further use this result to develop a semi-definite program that finds a maximal ellipsoid of provably attainable velocities. While these methods appear to provide faithful approximations, an additional simplification of the semi-definite program yields significantly faster under-approximations, at some expense of quality. A natural next step is to incorporate uncertainties in the unforced dynamics f and to consider cases with underactuated systems.

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