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Characterization of a Topological Obstruction to Reach Control by Continuous State Feedback

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Abstract This paper studies a topological obstruction to solving the Reach Control Problem (RCP) by continuous state feedback. Given a simplex and given an affine control system defined on the simplex, the RCP is to find a state feedback to drive closed-loop trajectories initiated in the simplex through an exit facet, without first exiting through other facets. We distill the problem as one of continuously extending a function that maps into a sphere from the boundary of a simplex to its interior. As such, we employ techniques from the extension problem of algebraic topology. Unlike previous work on the same problem, in this paper we remove unnecessary restrictions on the dimension of the simplex, the number of inputs of the system, and the particular geometry of the subset of the state space where the obstruction arises. Thus, the results of this paper represent the culmination of our efforts to characterize the topological obstruction. The conditions obtained in the paper are easily checkable and fully characterize the obstruction.

Keywords Reach Control Problem · Topological obstruction · Extension problem · Continuous state feedback

1 Introduction

This paper considers one of the open problems regarding the Reach Control Problem (RCP). We are given an affine control system whose state space is a (possibly non-convex) polytopal set which is assumed to have been triangulated. The RCP is to find a feedback control to drive closed-loop trajectories of the system through an exit facet of a particular simplex, without first exiting other facets. The central theoretical question regarding solvability of the RCP is to identify a suitable class of feedbacks (affine, multi-affine, time-varying, dynamic compensation, etc) that is sufficient to solve

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the RCP when it is solvable by open-loop controls. This inquiry has yielded a rich set of findings, paralleling analogous investigations on the feedback stabilization problem for nonlinear systems.

One of the fundamental questions regarding the RCP is to determine whether continuous state feedbacks are a sufficiently large class to solve the RCP when it is solvable by open loop controls. We give some history on this question. In [15], affine feedbacks were identified as the most plausible class of feedbacks to solve the problem. The modern formulation of the RCP appeared in [14, 29], and again affine feedbacks were the focus — they play the role of linear state feedbacks for the analogous stabilization problem of linear systems. Whether affine feedbacks were a sufficient class of feedbacks to fully solve the RCP was not known in 2006. In 2010, inspired by geometric control theory, a triangulation of the state space was selected for which it was shown that affine feedback and continuous state feedback are equivalent with respect to solvability of the RCP [6]. This special triangulation came about by identifying the affine space $\mathcal{O}$ where closed-loop equilibria could appear for an affine control system, and then imposing that $\mathcal{O}$ could only intersect simplices of the triangulated state space along faces of simplices. The result gave a satisfying confirmation of the relevance of earlier investigations on affine feedbacks, but it left open three questions. First, under the special triangulation, if the problem is not solvable by continuous state feedback, then is it solvable by any feedback? Second, if we do not impose the special triangulation, does the equivalence of affine feedback and continuous state feedback still hold? Third, what is the intrinsic reason why continuous state feedbacks fail?

The first question was fully answered in [7] where it was shown that a class of feedbacks that solves the RCP when it is solvable by open-loop controls is (discontinuous) piecewise affine feedbacks. Similar results using time-varying affine feedback appeared in [2]. The second question was resolved in [16] for a triangulation which is the same one studied in this paper. It was shown via an example that continuous state feedback and affine feedback are no longer equivalent for solving the RCP. Also in [16] it was shown that for single input systems, multi-affine feedbacks could solve the RCP even when affine feedbacks fail. The third question was never fully addressed, and it is the subject of this paper.

In [6] the failure of continuous state feedback to solve the RCP was tied to the appearance of closed-loop equilibria which could not be "pushed off" the simplex by a suitable feedback. Effectively, this was the first indication of a topological obstruction to solving the RCP by continuous state feedback. The terminology *topological obstruction* refers to an obstruction to extending a continuous map from the boundary of a domain to its interior. Thus, it belongs to the class of *extension problems* of algebraic topology. The paper [6] did not approach the problem from a topological perspective, though techniques from fixed point theory were employed. The problem was first formally recognized to regard a topological obstruction in [19], where some preliminary work was done. The first rigorous attempt at the problem appeared in [25] where necessary and sufficient conditions for the appearance of a topological obstruction on two and three dimensional simplices were presented. Further techniques were exploited in [24] in order to tackle the problem for the case of systems with two inputs. Additionally, [8] resolves the problem in the case that the possible set of equilibria $\mathcal{O}_{\mathcal{S}}$ of the control system has a specific geometric structure. Despite this progress, a complete characterization of the topological obstruction has, up to now, not been available.

The results of this paper are significantly more far-reaching than those of the above mentioned papers and represent the culmination of our effort to describe the topological obstruction in the RCP. Unlike [25], [24] and [8], the contributions of this paper do not rely on limiting the dimension of the state space, the number of inputs, or on a restrictive assumption on the structure of the set of possible equilibria (other than that set forming a simplex). While some technical assumptions remain and are discussed in the paper, the results of this paper largely solve the problem of a topological obstruction to solving the RCP by continuous state feedback.

The problem investigated in this paper is related to a similar problem of an affine obstruction; that is, an obstruction to solving the RCP using affine feedback. This problem was introduced in [25] and its supplementary paper [23], and solved there for two- and three-dimensional simplices. It has recently been solved in its entirety in [20]. Although the two problems of a topological obstruction and an affine obstruction are similar, the methods used to solve them are vastly different. The affine obstruction is resolved using linear algebra, whereas in this paper, the topological obstruction is tackled using methods of algebraic topology.

Our approach is as follows. We have a simplex $\mathcal{S}$ and an affine control system defined on $\mathcal{S}$. To solve the RCP we must guarantee trajectories do not exit facets which are not the exit facet. This is done by imposing certain inward pointing conditions, called invariance conditions, on the velocity vectors on the boundary of the simplex. Second, we must guarantee there are no closed-loop equilibria. The set of possible equilibria of an affine control system is well known to be an affine space $\mathcal{O}$. Thus the focus is on when it is possible to assign a continuous, non-vanishing map on $\mathcal{O}_{\mathcal{S}} := \mathcal{S} \cap \mathcal{O}$, while also meeting the invariance conditions. If we assume $\mathcal{O}_{\mathcal{S}}$ is a simplex, then the topological obstruction problem deals with determining whether there exists a continuous map from a simplex to a sphere, with constraints on the map at the faces of the simplex. The invariance conditions are such that the constraints get weaker as the dimension of the simplex face grows, i.e., the function values are most constrained on the vertices of $\mathcal{O}_{\mathcal{S}}$, then less constrained on the edges, and so on. This closely relates to the framework of Eilenberg obstruction theory [10, 12], and we modify its topological approach to fit our needs. We attempt to find a continuous function from a simplex to a sphere satisfying all the constraints by an inductive approach. The key to our strategy is as follows: in the first step, we choose function values on the vertices of the simplex. Then, by using the theory of absolute retracts, we attempt to extend this function to less constrained 1-dimensional faces, and continue onwards. Finally, after extending the function to the entire boundary of the simplex, we use the theory of null-homotopic functions to fill in the function in the simplex interior, where there are no constraints at all.

We can only perform the last step in the above strategy if the boundary map is null-homotopic. We use the known fact that non-surjective functions on a sphere are null-homotopic. On the other hand, if the constraints are such that the union of their interiors covers the entire codomain, we use a Sperner's lemma argument to show that it is impossible to obtain a continuous function on the whole simplex satisfying all of these constraints.

The paper is organized as follows. In Section 2 we introduce the topological tools to be used in our work on the obstruction problem. In Section 3 we define the Reach Control Problem, motivate and formally describe the problem of a topological obstruction, and convert it into the form most apt for our approach to the solution. Section 4 contains the list of assumptions present in this work, as well as several introductory results on the geometric properties of $\mathcal{O}_{\mathcal{S}}$ and other objects at hand. Section 5 and Section 6 contain the bulk of the novel work presented in this paper. In Section 5 we use extension theory to provide an elegant sufficient condition for the existence of a function solving the problem of a topological obstruction. In Section 6 we use Sperner's lemma to show that this sufficient condition, slightly strengthened, is also a necessary one. Section 7 contains a discussion on the relation of the results presented in this paper with the work of previous papers dealing with topological obstructions in RCP. We also discuss the modification of our approach to a triangulation of the state space different from the one present in the rest of the paper. In Section 8 we give concluding remarks and outline several open questions remaining on this topic. Finally, additional technical claims which are used in the paper, but deal with the problem of a topological obstruction only tangentially are proved in the Appendix.

Notation: The notation $\mathbf{0}$ refers to the subset of $\mathbb{R}^n$ containing only the zero element. Symbol $\mathbb{B}^n$ denotes the closed unit ball in $\mathbb{R}^n$ centered at the origin and $\mathbb{S}^{n-1} = \partial\mathbb{B}^n$ denotes the unit sphere in $\mathbb{R}^n$. If $\mathcal{A} \subset \mathbb{R}^n$ is a set, then $\partial\mathcal{A}$ denotes its (relative) boundary, $\overline{\mathcal{A}}$ denotes its closure, $\mathrm{int}(\mathcal{A})$ denotes its (relative) interior, and $\mathcal{A}^c$ denotes its complement. The notation $\mathrm{co}\{v_1, \ldots, v_k\}$ denotes the convex hull of the points $v_1, \ldots, v_k \in \mathbb{R}^n$, whereas $\mathrm{sp}\{v_1, \ldots, v_k\}$ denotes the vector subspace spanned by these points. If $F : \mathcal{X} \to \mathcal{Y}$ is a map whose domain $\mathcal{X}$ is a closed set, then the notation $\partial F : \partial\mathcal{X} \to \mathcal{Y}$ refers to the *boundary map* which coincides with F on the boundary of $\mathcal{X}$.

2 Background

We say two topological spaces $\mathcal{X}$ and $\mathcal{Y}$ are *homeomorphic* (notation $\mathcal{X} \cong \mathcal{Y}$) if there exists a continuous function $f : \mathcal{X} \to \mathcal{Y}$ which is bijective and has a continuous inverse. Such an f is called a *homeomorphism*. A topological space $\mathcal{X}$ is *locally contractible* if for every $x \in \mathcal{X}$ and every open neighborhood $\mathcal{V} \subseteq \mathcal{X}$ of x, there exists an open neighborhood $\mathcal{W} \subseteq \mathcal{V}$ of x which is contractible in the subspace topology from $\mathcal{V}$. The *k-th skeleton* of a polytope $\mathcal{P}$, denoted by $\partial^{(k)}\mathcal{P}$, is the union of all k-dimensional faces of $\mathcal{P}$.

Theorem 1 **([5])** *Let $\mathcal{P}$ be a convex κ-dimensional polytope. Then $\mathcal{P} \cong \mathbb{B}^\kappa$, and $\partial\mathcal{P} \cong \mathbb{S}^{\kappa-1} = \partial\mathbb{B}^\kappa$.*

Theorem 2 **(Theorem 1.4, [21])** *Let $\mathcal{X} \subseteq \mathbb{R}^m$ be convex, compact and non-empty. Then, $\mathcal{X} \cong \mathbb{B}^\rho$ for some $\rho \le m$.*

Lemma 1 **([1,4])** *A finite union of convex sets is locally contractible.*

We require several results from algebraic topology, particularly the extension problem and null-homotopic maps. The *extension problem* regards the following question: given a continuous map $\partial f : \partial\mathcal{X} \to \mathcal{Y}$ defined on the boundary of a space $\mathcal{X}$, we would like to know if there exists a continuous extension of $f : \mathcal{X} \to \mathcal{Y}$ such that $f|_{\partial\mathcal{X}} \equiv \partial f$. The terminology *topological obstruction* particularly refers to an obstruction to extending a continuous map. Let $f, g : \mathcal{X} \to \mathcal{Y}$ be continuous maps. We say f is *homotopic* to g, denoted by $f \simeq g$, if there exists a continuous function $F : \mathcal{X} \times [0,1] \to \mathcal{Y}$ such that $F(\cdot, 0) \equiv f$ and $F(\cdot, 1) \equiv g$. Topological spaces $\mathcal{X}$ and $\mathcal{Y}$ are *homotopy equivalent*, denoted $\mathcal{X} \simeq \mathcal{Y}$, if there exist continuous maps $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Y} \to \mathcal{X}$ such that $f \circ g$ and $g \circ f$ are homotopic to $id_{\mathcal{Y}}$ and $id_{\mathcal{X}}$, respectively. A continuous map $f : \mathcal{X} \to \mathcal{Y}$ is said to be *inessential* or *null-homotopic* if f is homotopic to a constant map $c(x) = y_0$, where $y_0 \in \mathcal{Y}$. A topological space $\mathcal{X}$ is *contractible* if the identity map $id_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}$ is null-homotopic.

Lemma 2 **([11])** *If $f : \mathcal{X} \to \mathbb{S}^n$ is a continuous map such that $f(\mathcal{X}) \neq \mathbb{S}^n$, that is, f is not surjective, then f is null-homotopic.*

The main result relating null-homotopic maps and the extension problem is the following important Extension Theorem (see [22]).

Theorem 3 *A continuous map $f : \mathbb{S}^n \to \mathcal{Y}$ is null-homotopic if and only if f extends to a continuous map $F : \mathbb{B}^{n+1} \to \mathcal{Y}$.*

The following definition of absolute retract, which also resembles the definition of absolute extensor, is taken from [1]. It may be noted that in metrizable spaces, the notions of absolute retract and absolute extensor (AE) are equivalent [26].

Definition 1 A metrizable space $\mathcal{X}$ is an *absolute retract* (AR) if for every metrizable space $\mathcal{Y}$ and every closed $\mathcal{V} \subseteq \mathcal{Y}$, each continuous map $f : \mathcal{V} \to \mathcal{X}$ is extendable to a continuous map $F : \mathcal{Y} \to \mathcal{X}$.

Definition 2 **([27])** A compact metric space is *finite-dimensional* if it is homeomorphic to a subset of $\mathbb{R}^k$ for some $k \in \mathbb{N}$.

Theorem 4 **([1,4])** *If $\mathcal{X}$ is a compact, contractible and locally contractible finite-dimensional metric space, it is AR.*

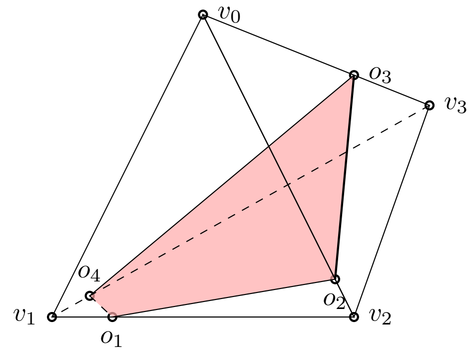

Fig. 1 3-dimensional simplex $\mathcal{S}$ with a possible set of potential equilibria $\mathcal{O}_{\mathcal{S}}$.

3 Reach Control Problem

We consider an $n-$dimensional simplex $\mathcal{S} := \text{co}\{v_0, v_1, \cdots, v_n\}$ with vertex set $V_{\mathcal{S}} := \{v_0, v_1, \cdots, v_n\}$ and facets $\mathcal{F}_0, \cdots, \mathcal{F}_n$ (the facet is indexed by the vertex it does not contain). Let h_i, $i \in \{0, \cdots, n\}$, be the unit normal vector to the facet $\mathcal{F}_i$ pointing outside the simplex, and let $\mathcal{F}_0$ be the *exit facet*. Define $I := \{1, \cdots, n\}$, and for $x \in \mathcal{S}$, let $I(x)$ be the minimal index set such that $x \in \text{co}\{v_i \mid i \in I(x)\}$.

Consider the affine control system defined on $\mathcal{S}$:

$$\dot{x} = Ax + Bu + a\,, \qquad\qquad x \in \mathcal{S}\,, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $a \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times m}$, and $\text{rank}(B) = m$. Let $\mathcal{B} = \text{Im}(B)$, the image of B. Also, let $\phi_u(t, x_0)$ denote the trajectory of (1) under a control law u starting from $x_0 \in \mathcal{S}$. We are interested in studying reachability of the exit facet $\mathcal{F}_0$ while the state is constrained in $\mathcal{S}$.

Problem 1 (Reach Control Problem (RCP)) Consider system (1) defined on a simplex $\mathcal{S}$. Find a continuous state feedback $u(x)$ such that for every $x_0 \in \mathcal{S}$, there exist $T \geq 0$ and $\gamma > 0$ such that

(i) $\phi_u(t, x_0) \in \mathcal{S}$ for all $0 \leq t \in [0, T]$.
(ii) $\phi_u(T, x_0) \in \mathcal{F}_0$.
(iii) $\phi_u(t, x_0) \notin \mathcal{S}$ for all $t \in (T, T + \gamma)$.

It is clear that a necessary condition for $u(x)$ to solve the RCP is that there are no closed-loop equilibria in $\mathcal{S}$, i.e. $Ax + Bu(x) + a \neq 0$, $x \in \mathcal{S}$. Closed-loop equilibria of (1) can only appear in the affine space $\mathcal{O} := \{x \in \mathbb{R}^n | Ax + a \in \mathcal{B}\}$ [29,6]. Thus, we are interested in equilibria in the set

$$\mathcal{O}_{\mathcal{S}} := \mathcal{S} \cap \mathcal{O}.$$

Since $\mathcal{O}$ is an affine space, either $\mathcal{O}_{\mathcal{S}} = \emptyset$ or $\mathcal{O}_{\mathcal{S}}$ is a convex polytope in $\mathcal{S}$ with a dimension κ with $0 \leq \kappa \leq n$. An example of $\mathcal{O}_{\mathcal{S}}$ is shown in Figure 1.

A second necessary condition for solving the RCP is that velocity vectors $Ax + Bu(x) + a$ must point inward at points in the facets $\mathcal{F}_i$, $i \in I$ [15]. To formalize this requirement, for $x \in \mathcal{S}$, define the closed, convex cone

$$\mathcal{C}(x) := \left\{\; y \in \mathbb{R}^n \mid h_j \cdot y \leq 0,\; j \in I \setminus I(x) \;\right\}.$$

where the convention is that if $I \setminus I(x) = \emptyset$, $\mathcal{C}(x) = \mathbb{R}^n$. Figure 2 illustrates the cones $\mathcal{C}(x)$ as shaded cones attached at various $x \in \mathcal{S}$ since they are used to characterize tangent velocity vectors. Notice

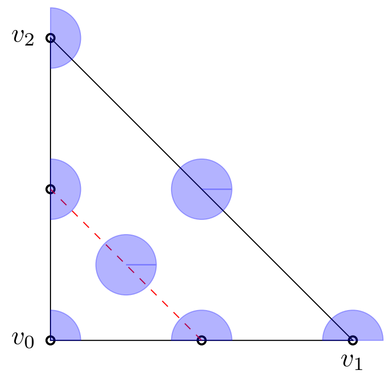

Fig. 2 Illustration of $\mathcal{C}(x)$, depicted as blue cones at several points $x \in \mathcal{S}$. An example of $\mathcal{O}_\mathcal{S}$ is depicted by a dashed line.

that for $x \in \mathcal{S} \setminus \mathcal{F}_0$, $\mathcal{C}(x)$ is exactly the Bouligand tangent cone to $\mathcal{S}$ at v_0, $T_\mathcal{S}(x)$. At $x \in \mathcal{F}_0$, $\mathcal{C}(x)$ includes directions pointing out of $\mathcal{F}_0$. The requirement that velocity vectors must point inwards can be formally stated as

$$Ax + Bu(x) + a \in \mathcal{C}(x), \quad x \in \mathcal{S}\,. \tag{2}$$

Let $u(x) : \mathcal{S} \to \mathbb{R}^m$ be a continuous state feedback. Define $f(x) = Ax + Bu(x) + a$. Notice that, by the definition of $\mathcal{O}_\mathcal{S}$, $f(x) \in \mathcal{B}$ for $x \in \mathcal{O}_\mathcal{S}$. From the foregoing discussion, a necessary condition for $u(x)$ to solve the RCP is that there exists $f : \mathcal{O}_\mathcal{S} \to \mathcal{B} \setminus \{0\}$ such that $f(x) \in \mathcal{C}(x)$, $x \in \mathcal{O}_\mathcal{S}$. We now transform this statement so that explicit consideration of $\mathcal{B}$ can be avoided; see also [20] where the same technique was used.

Since $\mathcal{B} \subset \mathbb{R}^n$ is an m-dimensional subspace, we can identify it with $\mathbb{R}^m$ through a linear transformation $M \in \mathbb{R}^{n\times m}$ whose columns form an orthonormal basis of $\mathcal{B}$, such that $M^TM = I$ and $MM^T|_\mathcal{B} \equiv id$. This interprets the function $f : \mathcal{O}_\mathcal{S} \to \mathcal{B}\backslash\{0\}$ as $\frac{M^T f}{\|M^T f\|} = \tilde{f} : \mathcal{O}_\mathcal{S} \to \mathbb{S}^{m-1}$. The requirement $f(x) \in \mathcal{C}(x)$ says $h_j \cdot f(x) \leq 0$, $j \in I \setminus I(x)$. Since $MM^T|_\mathcal{B} \equiv id$, and $f(x) \in \mathcal{B}$ for $x \in \mathcal{O}_\mathcal{S}, h_j \cdot f(x) = h_j \cdot MM^T f(x)$. Let $\tilde{h}_j := M^T h_j \in \mathbb{R}^m$. Then $f(x) \in \mathcal{C}(x)$ is equivalent to $\tilde{h}_j \cdot \tilde{f}(x) \leq 0$, $j \in I \setminus I(x)$.

Under the foregoing transformation, the problem studied in the paper is to determine if there exists a continuous function $\tilde{f} : \mathcal{O}_\mathcal{S} \to \mathbb{S}^{m-1}$ satisfying $\tilde{f}(x) \in \tilde{C}(x)$, $x \in \mathcal{O}_\mathcal{S}$, where $\tilde{C}(x) := \{y \in \mathbb{S}^{m-1} \mid \tilde{h}_j \cdot y \leq 0,\ j \in I \setminus I(x)\ \}$. In what follows we abuse notation and remove the tilde's from the variables f, h_j, and $\mathcal{C}(x)$. We arrive at the main problem studied in this paper.

Problem 2 Does there exist a continuous function $f : \mathcal{O}_\mathcal{S} \to \mathbb{S}^{m-1}$ satisfying

$$f(x) \in \mathcal{C}(x)\,, \qquad x \in \mathcal{O}_\mathcal{S}\,, \tag{3}$$

where

$$\mathcal{C}(x) := \left\{\ y \in \mathbb{S}^{m-1} \mid h_j \cdot y \leq 0,\ j \in I \setminus I(x)\ \right\}? \tag{4}$$

4 Preliminaries

In this section we define additional helpful notation and state the main assumptions for the problem. First we define the cones

$$\mathcal{C}_j = \left\{y \in \mathbb{S}^{m-1} \mid h_j \cdot y \leq 0\right\}, \quad j \in I\,. \tag{5}$$

Notice that $\mathcal{C}(x) = \cap_{j\in I\setminus I(x)}\mathcal{C}_j$, $x \in \mathcal{O}_\mathcal{S}$. We now introduce our main assumptions.

Assumption 1

(A1) *The pair* (A, B) *is controllable.*
(A2) $2 \leq m \leq n-1$.
(A3) *For any non-empty index set* $I' \subset I$, *if* $\mathcal{Y} := \bigcap_{j \in I'} \mathcal{C}_j \neq \emptyset$, *then* $\mathcal{Y} \cong \mathbb{B}^\rho$ *for some* $\rho \in \{0, \ldots, m-1\}$.
(A4) $\mathcal{O}_\mathcal{S} = \mathrm{co}\{o_1, \ldots, o_{\kappa+1}\}$, *a* κ*-dimensional simplex with vertices* $o_1, \ldots, o_{\kappa+1}$.
(A5) $v_0 \notin \mathcal{O}_\mathcal{S}$.
(A6) $\mathcal{O}_\mathcal{S} \cap \mathrm{int}(\mathcal{S}) \neq \emptyset$.
(A7) $\mathcal{C}(o_i) \neq \emptyset$, $i = 1, \ldots, \kappa+1$.

Assumption (A1) implies that $\dim(\mathcal{O}) = m$; see Lemma 3(i). Regarding Assumption (A2), we do not consider the case $m = 1$ because it was resolved by Theorem 1 in [30]; see also Section 7. Also we do not consider the case $m = n$, since then there is a trivial solution to the problem. Assumption (A3) is a non-degeneracy assumption which is discussed in greater detail below. Assumptions (A4)-(A6) regard the interaction between the simplex $\mathcal{S}$ and the set $\mathcal{O}$. This interaction arises from the choice of triangulation of the state space and is therefore under the designer's discretion; see also [30]. Assumption (A7) incorporates a necessary condition for solvability of the RCP: if there exists $f : \mathcal{O}_\mathcal{S} \to \mathbb{S}^{m-1}$ satisfying $f(x) \in \mathcal{C}(x)$ for all $x \in \mathcal{O}_\mathcal{S}$, then $f(o_i) \in \mathcal{C}(o_i)$. We enumerate the relevant known results under Assumption 1; see [30, 23].

Lemma 3

(i) *If* (A, B) *is controllable and* $\mathcal{O}_\mathcal{S} \cap \mathrm{int}(\mathcal{S}) \neq \emptyset$, *then* $\dim(\mathcal{O}_\mathcal{S}) = m$.
(ii) *If* $\dim(\mathcal{O}_\mathcal{S}) \geq 1$, *then* $\partial \mathcal{O}_\mathcal{S} \subseteq \partial \mathcal{S}$.
(iii) *Let* $\mathcal{P} \subseteq \mathcal{O}_\mathcal{S}$ *be a polytope and let* $x \in \mathrm{int}(\mathcal{P})$. *Then* $\mathcal{C}(y) \subseteq \mathcal{C}(x)$ *for all* $y \in \partial \mathcal{P}$.

Now we discuss (A3). Assumption (A3) ensures that non-empty intersections of the $\mathcal{C}_j$'s form contractible sets. This guarantees that the $\mathcal{C}_j$'s form good closed covers of the spaces we are observing; a similar condition was imposed in [24]. Developing the theory of topological obstructions in the RCP without this assumption is possible, but it results in a number of degenerate cases. In particular, (A3) ensures that each $\mathcal{C}_j$ must be contractible, so each $\mathcal{C}_j \neq \mathbb{S}^{m-1}$, which, in turn, implies each $h_j \neq 0$, $j \in I$. Since the cones $\mathcal{C}_j$ are projections of $\mathcal{B} \cap \mathcal{C}(x)$, $x \in \mathcal{F}_j$, onto $\mathcal{B}$ in the original set-up, this non-degeneracy assumption imposes constraints on the interaction between $\mathcal{B}$ and $\mathcal{S}$. It is possible to test whether (A3) is satisfied by a numerical procedure. First we have the following characterization of (A3).

Lemma 4 *Let* $I' \subset I$ *be a non-empty index set. Define* $\mathcal{Y} := \bigcap_{j \in I'} \mathcal{C}_j$ *and suppose* $\mathcal{Y} \neq \emptyset$. *Additionally, assume that there exists* $x \in \mathcal{Y}$ *such that* $h_j \cdot x < 0$ *for some* $j \in I'$. *Then,* $\mathcal{Y} \cong \mathbb{B}^\rho$ *for some* $\rho \in \{0, \ldots, m-1\}$.

Proof Clearly $-x \notin \mathcal{Y}$. Thus, we can use a stereographic projection centered at $-x$ to homeomorphically project $\mathcal{Y}$ on $\mathbb{R}^{m-1}$. By Lemma 11, this projection morphs $\mathcal{Y}$ into an intersection of closed balls and half-spaces, at least one of which, corresponding to $\mathcal{C}_j$, is a ball. Hence, the projection of $\mathcal{Y}$ on $\mathbb{R}^{m-1}$ is: closed, as an finite intersection of closed sets; bounded, as a subset of the ball corresponding to $\mathcal{C}_j$; convex, as an intersection of convex sets. By Theorem 2, $\mathcal{Y}$ is homeomorphic to a ball of some dimension $0 \leq \rho \leq m-1$. □

Suppose that $I' = \{j_1, \ldots, j_p\}$ in the previous lemma. Then the statement that there exists $x \in \mathcal{Y}$ such that $h_j \cdot x < 0$ for some $j \in I'$ is equivalent to the following: there exists an $x \in \mathbb{R}^m$ which is a solution to

$$\begin{bmatrix} h_{j_1}^T \\ h_{j_2}^T \\ \vdots \\ h_{j_p}^T \\ h_{j_1}^T + \ldots + h_{j_p}^T \end{bmatrix} x \leq \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix} . \tag{6}$$

This is a standard linear feasibility problem. We also have a converse statement to Lemma 4.

Lemma 5 *Suppose (A3) holds and let $I' \subset I$ be a non-empty index set. Define $\mathcal{Y} := \bigcap_{j \in I'} \mathcal{C}_j$ and suppose $\mathcal{Y} \neq \emptyset$. Then there exists $x \in \mathcal{Y}$ such that $h_j \cdot x < 0$ for all $j \in I'$.*

Proof We prove the statement by induction on $|I'|$, the cardinality of I'. Consider $I' \subset I$ such that $|I'| = 1$ and $\mathcal{Y} := \cap_{j \in I'} \mathcal{C}_j \neq \emptyset$. Assumption (A3) implies $h_j \neq 0$, so there clearly exists $x \in \mathcal{Y}$ such that $h_j \cdot x < 0$, $j \in I'$. Now supppose the statement holds for all index sets with cardinality less than $k+1$. Consider any index set $I' \subset I$ such that $|I'| = k+1$ and $\mathcal{Y} := \cap_{j \in I'} \mathcal{C}_j \neq \emptyset$.

First we claim there exist $x \in \mathcal{Y}$ and $j' \in I'$ such that $h_{j'} \cdot x < 0$. For if not, then for all $x \in \mathcal{Y}$ and $j' \in I'$, $h_{j'} \cdot x = 0$. This implies $\mathcal{Y} \subset \{z \in \mathbb{S}^{m-1} \mid h_j \cdot z = 0, j \in I'\}$. But by definition of $\mathcal{Y}$, $\{z \in \mathbb{S}^{m-1} \mid h_j \cdot z = 0, j \in I'\} \subset \mathcal{Y}$, and thus $\mathcal{Y} = \{z \in \mathbb{S}^{m-1} \mid h_j \cdot z = 0, j \in I'\}$, the intersection of $(m-2)$-dimensional spheres. Hence, it is itself a sphere, which contradicts (A3).

Thus, there exist $x \in \mathcal{Y}$ and $j' \in I'$ such that $h_{j'} \cdot x < 0$. By the induction step there exists x' such that $h_j \cdot x' < 0$ for all $j \in I' \backslash \{j'\}$ (since $|I' \setminus \{j'\}| = k < k+1$). We observe that $x \neq -\lambda x'$ for any $\lambda > 0$ because $x \in \mathcal{Y}$ implies $h_j \cdot x \leq 0$ for all $j \in I'$. But if $x = -\lambda x'$, then $h_j \cdot (-\lambda x') = h_j \cdot x > 0$, $j \in I' \setminus \{j'\}$, a contradiction. Now consider $\tilde{x} = (x + \lambda x')/\|x + \lambda x'\|$. Since $h_j \cdot x \leq 0$ and $h_j \cdot x' < 0$ for $j \in I' \backslash \{j'\}$, we have $h_j \cdot \tilde{x} = \frac{1}{\|x+\lambda x'\|}\big(h_j \cdot x + h_j \cdot x'\big) < 0$ for $j \in I' \backslash \{j'\}$. Also since $h_{j'} \cdot x < 0$, we can choose λ sufficiently small such that $h_{j'} \cdot \tilde{x} < 0$. Thus, $\tilde{x} \in \mathcal{Y}$ satisfies the statement. □

In light of (A4), we define $I_{\mathcal{O}_\mathcal{S}} := \{1, \ldots, \kappa+1\}$, and we denote the facets of the simplex $\mathcal{O}_\mathcal{S}$ as $\mathcal{F}_k^{\mathcal{O}}$, $k = 1, \ldots, \kappa+1$, where $\mathcal{F}_k^{\mathcal{O}}$ is the facet not containing the vertex o_k. Consider any $\mathcal{F}_j^{\mathcal{O}}$. We observe that $I(x) = \cup_{i \in \{1,\ldots,j-1,j+1,\ldots,\kappa+1\}} I(o_i)$ for all $x \in \mathrm{int}(\mathcal{F}_j^{\mathcal{O}})$, so $\mathcal{C}(x)$ are the same for every $x \in \mathrm{int}(\mathcal{F}_j^{\mathcal{O}})$. Therefore we can define

$$\mathcal{H}_j := \mathcal{C}(x)\,, \quad x \in \mathrm{int}(\mathcal{F}_j^{\mathcal{O}}), \quad j \in I_{\mathcal{O}_\mathcal{S}}\,. \tag{7}$$

Notice that $\mathcal{H}_j$ is a closed subset in $\mathbb{S}^{m-1}$. Also define $\mathcal{H}_j^c := \mathbb{S}^{m-1} \setminus \mathcal{H}_j$, $j \in I_{\mathcal{O}_\mathcal{S}}$, which is an open subset in $\mathbb{S}^{m-1}$. We have the following fact about the sets $\mathcal{H}_j$.

Lemma 6 *Suppose (A3)-(A6) hold. At most one $\mathcal{H}_j$, $j \in I_{\mathcal{O}_\mathcal{S}}$, satisfies $\mathcal{H}_j = \mathbb{S}^{m-1}$.*

Proof Suppose not. First we take any $j \in I_{\mathcal{O}_\mathcal{S}}$ such that $\mathcal{H}_j = \mathbb{S}^{m-1}$ and $x \in \mathrm{int}(\mathcal{F}_j^{\mathcal{O}})$. By definition $\mathcal{H}_j = \bigcap_{i \in I \setminus I(x)} \mathcal{C}_i$. In order that $\mathcal{H}_j = \mathbb{S}^{m-1}$, either $\mathcal{C}_i = \mathbb{S}^{m-1}$ for all $i \in I \backslash I(x)$ or $I \setminus I(x) = \emptyset$. First, $\mathcal{C}_i = \mathbb{S}^{m-1}$ contradicts (A3). Second, suppose $I(x) = \{0, 1, \ldots, n\}$. This is impossible by Lemma 3(ii), since $x \in \mathcal{F}_j^{\mathcal{O}} \subset \partial \mathcal{O}_\mathcal{S} \subset \partial \mathcal{S}$. Third, suppose $I(x) = I$. Then $x \in \mathcal{F}_0$. Since $I \setminus I(z)$ is the same for all $z \in \mathrm{int}(\mathcal{F}_j^{\mathcal{O}})$, we deduce $\mathrm{int}(\mathcal{F}_j^{\mathcal{O}}) \subset \mathcal{F}_0$. Since $\mathcal{F}_j^{\mathcal{O}}$ and $\mathcal{F}_0$ are both polytopes, $\mathcal{F}_j^{\mathcal{O}} \subset \mathcal{F}_0$. Since $\mathcal{O}_\mathcal{S}$ is a simplex, $o_i \in \mathcal{F}_0$ for all $i \neq j$. Now we repeat this argument for $j' \neq j$ such that $\mathcal{H}_{j'} = \mathbb{S}^{m-1}$. Then $\mathcal{F}_{j'}^{\mathcal{O}} \subset \mathcal{F}_0$. Thus, $\mathcal{O}_\mathcal{S}$ is a simplex with two facets in $\mathcal{F}_0$, which implies $\mathcal{O}_\mathcal{S} \subset \mathcal{F}_0$. This contradicts (A6). □

Finally we define

$$\mathcal{H} := \bigcup_{\mathcal{H}_j \neq \mathbb{S}^{m-1}} \mathcal{H}_j\,, \qquad \mathcal{H}^\star := \bigcup_{\mathcal{H}_j \neq \mathbb{S}^{m-1}} \mathrm{int}(\mathcal{H}_j)\,. \tag{8}$$

5 Extension Problem

In this section we begin our study of Problem 2 by investigating when it is possible to extend a continuous boundary map ∂f from $\partial\mathcal{O}_\mathcal{S}$, the boundary of $\mathcal{O}_\mathcal{S}$, to its interior. Our main tool will be the Extension Theorem 3. Some supplemental claims which are intuitively clear, but necessary for our argument, are found in the Appendix.

The main idea is as follows. By (A7) we can construct a vertex map $f(o_i) \in \mathcal{C}(o_i)$, $i \in I_{\mathcal{O}_\mathcal{S}}$. We want to continuously extend this map to all of $\mathcal{O}_\mathcal{S}$ such that $f(x) \in \mathcal{C}(x)$, $x \in \mathcal{O}_\mathcal{S}$. We observe that if $\mathcal{C}(x) = \mathbb{S}^{m-1}$ for some $x \in \mathcal{O}_\mathcal{S}$, then the constraint $f(x) \in \mathcal{C}(x)$ is vacuous. For example, if $x \in \mathrm{int}(\mathcal{O}_\mathcal{S})$ then by Lemma 3(ii), $x \in \mathrm{int}(\mathcal{S})$, and $\mathcal{C}(x) = \mathbb{S}^{m-1}$. Thus, the only relevant constraints on $f(x)$ arise on points $x \in \partial\mathcal{O}_\mathcal{S}$, and moreover, points in $\partial\mathcal{O}_\mathcal{S}$ where $\mathcal{C}(x) \neq \mathbb{S}^{m-1}$. We can think of $\mathcal{H}$ as capturing the co-domain of a boundary map $\partial f : \partial\mathcal{O}_\mathcal{S} \to \mathcal{H}$ where the constraints are non-vacuous. If this co-domain does not cover all of $\mathbb{S}^{m-1}$, that is $\mathcal{H} \neq \mathbb{S}^{m-1}$, then the boundary map is not surjective and Lemma 2 and Theorem 3 allow us to argue that the boundary map can be extended to all of $\mathcal{O}_\mathcal{S}$. A key step in this construction is to extend a map $f : \partial\mathcal{P} \to \mathcal{H}$ from the boundary of a face $\mathcal{P}$ of $\mathcal{O}_\mathcal{S}$ to its interior. We do that in the following lemma by showing that the codomain of the map on $\mathcal{P}$, namely $\mathcal{Y} := \mathcal{C}(x) \cap \mathcal{H}$, $x \in \mathrm{int}(\mathcal{P})$, is AR. The main tool to establish that $\mathcal{Y}$ is an algebraic retract is Theorem 4. This requires showing that $\mathcal{Y}$ is compact, contractible, and locally contractible, which is done in Lemmas 12 and 13.

Lemma 7 *Suppose Assumption 1 holds, and suppose $\mathcal{H} \neq \mathbb{S}^{m-1}$. Let $\mathcal{P}$ be any k-dimensional face of $\mathcal{O}_\mathcal{S}$ with $0 \le k \le \kappa - 1$. Let $\mathcal{Y} := \mathcal{C}(x) \cap \mathcal{H}$ for any $x \in \mathrm{int}(\mathcal{P})$. Then, $\mathcal{Y}$ is AR.*

Proof We consider two cases. First suppose there exists a facet $\mathcal{F}_j^\mathcal{O}$ of $\mathcal{O}_\mathcal{S}$ with $\mathcal{P} \subset \mathcal{F}_j^\mathcal{O}$ such that $\mathcal{H}_j \neq \mathbb{S}^{m-1}$. By Lemma 3(iii), $\mathcal{C}(x) \subseteq \mathcal{H}_j \subseteq \mathcal{H}$. Since $\mathcal{Y} := \mathcal{C}(x) \cap \mathcal{H}$, we get $\mathcal{Y} = \mathcal{C}(x)$ for any $x \in \mathrm{int}(\mathcal{P})$. Further, since $\mathcal{C}(x) = \cap_{j \in I \setminus I(x)} \mathcal{C}_j$, by (A3), $\mathcal{Y} = \mathcal{C}(x)$ is homeomorphic either to a ball or a sphere, so, it is locally contractible. Also, by definition $\mathcal{Y}$ is compact. In sum, by Theorem 4, $\mathcal{Y}$ is AR.

Second, suppose $\mathcal{H}_j = \mathbb{S}^{m-1}$ for all facets $\mathcal{F}_j^\mathcal{O}$ of $\mathcal{O}_\mathcal{S}$ such that $\mathcal{P} \subseteq \mathcal{F}_j^\mathcal{O}$. By Lemma 6, there is at most one facet $\mathcal{F}_j^\mathcal{O}$ such that $\mathcal{H}_j = \mathbb{S}^{m-1}$. Thus, $\mathcal{P} = \mathcal{F}_j^\mathcal{O}$ with $\mathcal{H}_j = \mathbb{S}^{m-1}$, and $\mathcal{Y} = \mathcal{H}_j \cap \mathcal{H} = \mathbb{S}^{m-1} \cap \mathcal{H} = \mathcal{H}$. Recall that o_j is the vertex of $\mathcal{O}_\mathcal{S}$ not contained in $\mathcal{P}$. By Lemma 3(iii) and (7), $\mathcal{C}(o_j) \subseteq \mathcal{H}_i$ for all $i = 1, \ldots, j-1, j+1, \ldots, \kappa+1$. Then by (A7), $\bigcap_{\mathcal{H}_i \neq \mathbb{S}^{m-1}} \mathcal{H}_i \neq \emptyset$. By Lemma 13, $\mathcal{H}$ is contractible; by Lemma 12, $\mathcal{H}$ is locally contractible; and by definition, $\mathcal{H}$ is compact. In sum, by Theorem 4, $\mathcal{Y} = \mathcal{H}$ is AR. □

We now give the main result on extending vertex maps. Here is the roadmap for our proof strategy. We must show that if $\mathcal{H} \neq \mathbb{S}^{m-1}$, then there exists a continuous map $f : \mathcal{O}_\mathcal{S} \to \mathbb{S}^{m-1}$ satisfying the cone conditions (3). We employ an induction argument on the dimension of the faces of $\mathcal{O}_\mathcal{S}$. For the base step, we define a vertex map ∂f on the vertices $\{o_1, \ldots, o_{\kappa+1}\}$ of $\mathcal{O}_\mathcal{S}$ such that $\partial f(o_i) \in \mathcal{C}(o_i)$. To inductively build up ∂f on the faces of $\mathcal{O}_\mathcal{S}$, we assume a map with the required properties is defined on the k-th skeleton of $\mathcal{O}_\mathcal{S}$ consisting of all k-dimensional faces. This map is extended to the $(k+1)$-th skeleton by invoking Lemma 7 on each $(k+1)$-dimensional face $\mathcal{P}$. Specifically, the set $\mathcal{Y} := \mathcal{C}(x)$, $x \in \mathrm{int}(\mathcal{P})$, which is effectively the codomain of the map, is an algebraic retract. We continue with this procedure until we reach $\partial f : \partial\mathcal{O}_\mathcal{S} \to \mathbb{S}^{m-1}$ satisfying (3). Based on the assumption that the union of cones $\mathcal{C}(x)$, $x \in \partial\mathcal{O}_\mathcal{S}$, does not cover $\mathbb{S}^{m-1}$, the map ∂f is not surjective. Then, as mentioned above, Lemma 2 and Theorem 3 are used to conclude that ∂f can be continuously extended to the interior of $\mathcal{O}_\mathcal{S}$. Fortunately, the interior of $\mathcal{O}_\mathcal{S}$ lies in the interior of $\mathcal{S}$, where $\mathcal{C}(x) = \mathbb{S}^{m-1}$ and the conditions $f(x) \in \mathcal{C}(x)$ are trivially satisfied.

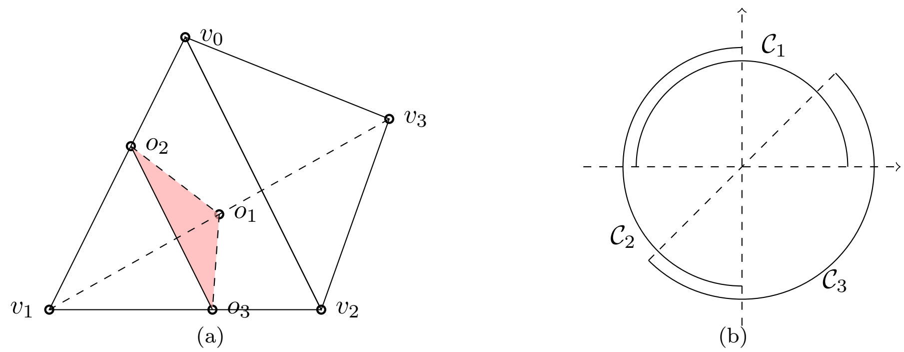

Fig. 3 Configuration of the set $\mathcal{O}_\mathcal{S}$ and the semi-circles $\mathcal{C}_i$ in Example 1.

Example 1 We illustrate the proof of the main result on extending continuous maps on $\mathcal{O}_\mathcal{S}$ via an example with $n = 3$, $m = 2$ and $\mathcal{O}_\mathcal{S}$ as in Figure 3. This example was already solved in two different ways in [25] and [24]; the figures are adapted from those papers.

From Figure 3 we observe that $\mathcal{H}_1 = \mathcal{C}_3$, $\mathcal{H}_2 = \mathbb{S}^1$ and $\mathcal{H}_3 = \mathcal{C}_2$. Then $\mathcal{H} := \bigcup_{\mathcal{H}_j \neq \mathbb{S}^1} \mathcal{H}_j \neq \mathbb{S}^1$. We can easily find a function $f : \mathcal{O}_\mathcal{S} \to \mathbb{S}^1$ satisfying all the cone conditions (3). Following the procedure of the proof of Theorem 5 given below, we first define a vertex map $\partial f(o_i)$, $i = 1, 2, 3$, such that $\partial f(o_1) \in \mathcal{C}_2$, $\partial f(o_2) \in \mathcal{C}_2 \cap \mathcal{C}_3 \neq \emptyset$, and $\partial f(o_3) \in \mathcal{C}_3$. Next, we continuously connect $\partial f(o_1)$ and $\partial f(o_2)$ by any curve in $\mathbb{S}^1$ which remains in $\mathcal{C}_2$. Similarly we continuously connect $\partial f(o_2)$ and $\partial f(o_3)$ by any curve in $\mathbb{S}^1$ which remains in $\mathcal{C}_3$. For instance, we can choose the shortest arcs between points $\partial f(o_1)$ and $\partial f(o_2)$, and points $\partial f(o_2)$ and $\partial f(o_3)$. As there are no restrictions on the edge between $\overline{o_1 o_3}$, we can connect these points by a curve that stays in $\mathcal{H} = \mathcal{C}_2 \cup \mathcal{C}_3$. These curves then determine a continuous map ∂f defined on $\partial \mathcal{O}_\mathcal{S}$ that satisfies all the cone conditions. Observe that the co-domain of this boundary map lies entirely in $\mathcal{H}$. Then because $\mathcal{H} \neq \mathbb{S}^1$, the boundary map is not surjective, so it is null-homotopic. Finally, by Theorem 3, it can be extended to a continuous function $f : \mathcal{O}_\mathcal{S} \to \mathbb{S}^1$. Such a function again satisfies all the cone conditions, because there are no new restrictions on the interior of $\mathcal{O}_\mathcal{S}$ because it lies in the interior of $\mathcal{S}$. ◁

Theorem 5 *Suppose Assumption 1 holds. Suppose $\mathcal{H} \neq \mathbb{S}^{m-1}$. Then there exists a continuous map $f : \mathcal{O}_\mathcal{S} \to \mathbb{S}^{m-1}$ such that $f(x) \in \mathcal{C}(x)$, $x \in \mathcal{O}_\mathcal{S}$.*

Proof The plan of the proof is, first, to construct a boundary map $\partial f : \partial \mathcal{O}_\mathcal{S} \to \mathcal{H}$ such that $\partial f(x) \in \mathcal{C}(x)$ for all $x \in \partial \mathcal{O}_\mathcal{S}$. Second, this map is extended to all of $\mathcal{O}_\mathcal{S}$. By (A7), $\mathcal{C}(o_i) \neq \emptyset$, $i \in I_{\mathcal{O}_\mathcal{S}}$, so take any $b_i \in \mathcal{C}(o_i)$ and defined $\partial f(o_i) := b_i$, $i \in I_{\mathcal{O}_\mathcal{S}}$. Each o_i lies in κ facets, where $\kappa = m \geq 2$, by (A2) and Lemma 3(i). By Lemma 6, at most one $j \in I_{\mathcal{O}_\mathcal{S}}$ satisfies $\mathcal{H}_j = \mathbb{S}^{m-1}$. Thus, $o_i \in \mathcal{F}_j^\mathcal{O}$ for some $j \in I_{\mathcal{O}_\mathcal{S}}$ such that $\mathcal{H}_j \neq \mathbb{S}^{m-1}$. By Lemma 3(iii), $\mathcal{C}(o_i) \subset \mathcal{H}_j \subset \mathcal{H}$.

We have defined ∂f on $\partial^{(0)} \mathcal{O}_\mathcal{S}$. We now define it on $\partial \mathcal{O}_\mathcal{S} = \partial^{(m-1)} \mathcal{O}_\mathcal{S}$ by inductively building up the map on $\partial^{(k)} \mathcal{O}_\mathcal{S}$. Assume that ∂f is defined on $\partial^{(k)} \mathcal{O}_\mathcal{S}$, and that it satisfies $\partial f(x) \in \mathcal{C}(x) \cap \mathcal{H}$ for all $x \in \partial^{(k)} \mathcal{O}_\mathcal{S}$. Take any $(k+1)$-dimensional face $\mathcal{P}$ of $\mathcal{O}_\mathcal{S}$, where $k + 1 < \kappa$. It suffices to show that ∂f can be extended on $\mathcal{P}$ such that $\partial f(x) \in \mathcal{C}(x) \cap \mathcal{H}$ for $x \in \text{int}(\mathcal{P})$. Define

$$\mathcal{Y} := \mathcal{C}(x) \cap \mathcal{H} \tag{9}$$

for some $x \in \text{int}(\mathcal{P})$. By Lemma 3(iii) and the definition of $\partial f(x)$, $\partial f(x) \in \mathcal{Y}$ for all $x \in \partial \mathcal{P}$. Thus, $\partial f|_{\partial \mathcal{P}} : \partial \mathcal{P} \to \mathcal{Y}$. By Lemma 7, $\mathcal{Y}$ is AR. Hence, the map $\partial f|_{\partial \mathcal{P}}$ can be extended to a map $\partial f : \mathcal{P} \to \mathcal{Y}$. Such a map satisfies $\partial f(x) \in \mathcal{Y} = \mathcal{C}(x) \cap \mathcal{H}$ for $x \in \text{int}(\mathcal{P})$, so we are hence done.

Now we have constructed $\partial f : \partial \mathcal{O}_\mathcal{S} \to \mathcal{H}$ such that $\partial f(x) \in \mathcal{C}(x)$, $x \in \partial \mathcal{O}_\mathcal{S}$. Next we interpret ∂f as a map into $\mathbb{S}^{m-1}$; that is, $\partial f : \partial \mathcal{O}_\mathcal{S} \to \mathbb{S}^{m-1}$. Since $\mathcal{H} \neq \mathbb{S}^{m-1}$, ∂f is not surjective. By Lemma 2,

$\partial f : \partial \mathcal{O}_{\mathcal{S}} \to \mathbb{S}^{m-1}$ is null-homotopic. By Lemma 3 and Theorem 1, $\partial \mathcal{O}_{\mathcal{S}} \cong \mathbb{S}^{m-1}$. By Theorem 3, ∂f extends to a map $f : \mathcal{O}_{\mathcal{S}} \cong \mathbb{B}^{m-1} \to \mathbb{S}^{m-1}$. Finally, since by construction $\partial f(x) \in \mathcal{C}(x)$ for all $x \in \partial \mathcal{O}_{\mathcal{S}}$, we have $f(x) = \partial f(x) \in \mathcal{C}(x)$ for $x \in \partial \mathcal{O}_{\mathcal{S}}$. For $x \in \text{int}(\mathcal{O}_{\mathcal{S}})$, by (A7) and Lemma 3, $x \in \text{int}(\mathcal{S})$. Thus, $\mathcal{C}(x) = \mathbb{S}^{m-1}$, and it immediately follows that $f(x) \in \mathcal{C}(x)$, $x \in \text{int}(\mathcal{O}_{\mathcal{S}})$. □

We note that the sufficient condition $\mathcal{H} \neq \mathbb{S}^{m-1}$, while elegant, is difficult to numerically verify. The set $\mathcal{H}$ is a union of finitely many sets $\mathcal{C}(x) = \{y \in \mathbb{S}^{m-1} \mid h_j \cdot y \leq 0, j \in I \backslash I(x)\}$. By defining $\mathcal{C}'(x) = \{\lambda y \mid \lambda \geq 0, y \in \mathcal{C}(x)\}$, the problem of verifying $\mathcal{H} \neq \mathbb{S}^{m-1}$ is equivalent to the problem of verifying whether a union of polyhedral cones $\mathcal{C}'(x) \subseteq \mathbb{R}^m$ covers the entire Euclidean space. As mentioned in [20], this is known to be an NP-complete problem (see, e.g., [13]).

6 An Obstruction

In this section we study when it is not possible to extend a continuous boundary map ∂f from $\partial \mathcal{O}_{\mathcal{S}}$, the boundary of $\mathcal{O}_{\mathcal{S}}$, to its interior. The condition that forces the obstruction is that $\mathcal{H}^\star$ covers $\mathbb{S}^{m-1}$. We note that this condition is almost the exact opposite of the sufficient condition for solvability in the previous section, namely $\mathcal{H} \neq \mathbb{S}^{m-1}$. The only gap arises when $\mathcal{H}^\star \neq \mathbb{S}^{m-1}$, but $\mathcal{H}$, the closure of $\mathcal{H}^\star$, equals $\mathbb{S}^{m-1}$. Thus, we arrive at necessary and sufficient conditions, up to a small gap due to a degenerate case.

Our main tool for the proof is Sperner's Lemma [3], which has been employed before to prove existence of closed-loop equilibria in the RCP [6]. The essential argument here follows by contradiction: suppose a continuous map $f : \mathcal{O}_{\mathcal{S}} \to \mathbb{S}^{m-1}$ exists such that $f(x) \in \mathcal{C}(x)$. Each vertex o_i of $\mathcal{O}_{\mathcal{S}}$ is assigned its own distinct "color" corresponding to whether $f(o_i) \notin \mathcal{H}_i$. Next, successively finer triangulations are constructed on $\mathcal{O}_{\mathcal{S}}$ following the rules of a proper labelling of colors to satisfy Sperner's lemma, as defined below. By Sperner's lemma, for each such triangulation, there exists a simplex of the triangulation whose vertex set includes all colors. Taking the limit as the diameter of the triangulations goes to zero, we deduce there exists a point $\overline{x}$ where $f(\overline{x}) \notin \mathcal{H}_i$, $i \in I_{\mathcal{O}_{\mathcal{S}}}$. This will contradict that $\mathcal{H}^\star$ covers $\mathbb{S}^{m-1}$.

We now present the setup of Sperner's lemma. Let $\mathbb{T}$ be a triangulation of an n-dimensional simplex $\mathcal{S}$. That is, $\mathbb{T}$ is a subdivision of $\mathcal{S}$ into n-dimensional simplices such that any two simplices of $\mathbb{T}$ intersect in a common face or not at all. A *proper labeling* of the vertices of $\mathbb{T}$ is as follows:

(L1) Each vertex of the original simplex $\mathcal{S}$ has its own distinct label.
(L2) Vertices of $\mathbb{T}$ on a face of $\mathcal{S}$ are labeled using only the labels of the vertices forming the face.

Given a properly labeled triangulation of $\mathcal{S}$, we say a simplex in $\mathbb{T}$ is *distinguished* if its vertices have all $n+1$ labels.

Lemma 8 (Sperner) *Every properly labeled triangulation of $\mathcal{S}$ has an odd number of distinguished simplices.*

Before applying Sperner's lemma to our problem, we need the following result which underlies our choice of a proper labelling.

Lemma 9 *Suppose Assumption 1 holds. Also suppose $\mathcal{H} = \mathbb{S}^{m-1}$. Then $\bigcap_{\mathcal{H}_j \neq \mathbb{S}^{m-1}} \mathcal{H}_j = \emptyset$.*

Proof Suppose by way of contradiction that $\mathcal{Y} := \bigcap_{\mathcal{H}_j \neq \mathbb{S}^{m-1}} \mathcal{H}_j \neq \emptyset$. Since each $\mathcal{H}_j$ is itself an intersection of $\mathcal{C}_k$'s, we can apply Lemma 5. That is, there exists $y \in \mathcal{Y} \subset \mathcal{H}$ such that $h_k \cdot y < 0$ for all $\mathcal{C}_k$'s contributing to $\mathcal{Y}$. Then $h_k \cdot (-y) > 0$ for all such $\mathcal{C}_k$'s so $-y \notin \mathcal{H}_j$ when $\mathcal{H}_j \neq \mathbb{S}^{m-1}$. This implies $-y \notin \mathcal{H}$, which contradicts that $\mathcal{H} = \mathbb{S}^{m-1}$. □

Theorem 6 *Suppose Assumption 1 holds. Also suppose $\mathcal{H}^\star = \mathbb{S}^{m-1}$. There does not exist a continuous map $f : \mathcal{O}_\mathcal{S} \to \mathbb{S}^{m-1}$ such that $f(x) \in \mathcal{C}(x)$, $x \in \mathcal{O}_\mathcal{S}$.*

Proof First we note that if $\mathcal{H}^\star = \mathbb{S}^{m-1}$, then $\mathcal{H} = \mathbb{S}^{m-1}$. Suppose by way of contradiction there exists a continuous map $f : \mathcal{O}_\mathcal{S} \to \mathbb{S}^{m-1} = \mathcal{H}$ such that $f(x) \in \mathcal{C}(x)$, $x \in \mathcal{O}_\mathcal{S}$. Suppose that, after possibly reordering indices, $\mathcal{H}_1 = \mathbb{S}^{m-1}$. By our index convention, $o_1 \in \mathcal{F}_j^\mathcal{O}$, $j \in I_{\mathcal{O}_\mathcal{S}} \setminus \{1\}$, so by Lemma 3(iii), $\mathcal{C}(o_1) \subset \mathcal{H}_j$, $j \in I_{\mathcal{O}_\mathcal{S}} \setminus \{1\}$. Then since $f(o_1) \in \mathcal{C}(o_1)$, we have that $f(o_1) \in \mathcal{H}_j$, $j \in I_{\mathcal{O}_\mathcal{S}} \setminus \{1\}$. Then $f(o_1) \in \cap_{\mathcal{H}_j \neq \mathbb{S}^{m-1}} \mathcal{H}_j$, which contradicts Lemma 9. We deduce that $\mathcal{H}_j \neq \mathbb{S}^{m-1}$, $j \in I_{\mathcal{O}_\mathcal{S}}$. This implies $\mathcal{H} = \bigcup_{j \in I_{\mathcal{O}_\mathcal{S}}} \mathcal{H}_j$ and $\mathcal{H}^\star = \bigcup_{j \in I_{\mathcal{O}_\mathcal{S}}} \mathrm{int}(\mathcal{H}_j)$. Now we apply Sperner's lemma. The first step is to obtain a proper labeling of $\mathcal{O}_\mathcal{S}$. We define the sets

$$\mathcal{Q}_i := f^{-1}(\mathcal{H}_i^c)\,, \qquad i \in I_{\mathcal{O}_\mathcal{S}}\,.$$

Since $f : \mathcal{O}_\mathcal{S} \to \mathbb{S}^{m-1}$ is continuous and $\mathcal{H}_i^c$ is open, each $\mathcal{Q}_i$ is an open subset of $\mathcal{O}_\mathcal{S}$. Because $f(x) \in \mathcal{C}(x)$, $x \in \mathcal{O}_\mathcal{S}$, we observe that

$$f(o_i) \in \mathcal{H}_j\,, \quad i \in I_{\mathcal{O}_\mathcal{S}}\,, j \in I_{\mathcal{O}_\mathcal{S}} \setminus \{i\}\,. \tag{10}$$

From (10) it is immediate that $o_i \notin \mathcal{Q}_j$ when $i \neq j$. Also, $o_i \in \mathcal{Q}_i$, for otherwise, invoking (10), we would have $f(o_i) \in \cap_{j \in I_{\mathcal{O}_\mathcal{S}}} \mathcal{H}_j$, which contradicts Lemma 9. Thus, inclusion in a set $\mathcal{Q}_i$ provides a distinct label for the vertices o_i of $\mathcal{O}_\mathcal{S}$. This satisfies (L1) of a proper labeling of $\mathcal{O}_\mathcal{S}$. Next, let $\mathbb{T}$ be any triangulation of $\mathcal{O}_\mathcal{S}$ and consider a vertex v of $\mathbb{T}$ which is not a vertex of $\mathcal{O}_\mathcal{S}$ and lies in $\partial\mathcal{O}_\mathcal{S}$. Without loss of generality, let $v \in \mathrm{co}\{o_1, \ldots, o_l\}$ for some $2 \leq l \leq \kappa$. Then it must be that $v \in \mathcal{Q}_k$ for some $1 \leq k \leq l$. For suppose not, that is $f(v) \in \mathcal{H}_j$, $j = 1, \ldots, l$. Since $v \in \mathrm{co}\{o_1, \ldots, o_l\}$, $v \in \mathcal{F}_{l+1}^\mathcal{O} \cap \cdots \cap \mathcal{F}_{\kappa+1}^\mathcal{O}$, which means $f(v) \in \mathcal{H}_{l+1} \cap \ldots \cap \mathcal{H}_{\kappa+1}$. In sum, $f(v) \in \cap_{j \in I_{\mathcal{O}_\mathcal{S}}} \mathcal{H}_j$, which contradicts Lemma 9. Clearly this labeling of v satisfies the second condition (L2) for a proper labeling. Finally, for vertices v of $\mathbb{T}$ in the interior of $\mathcal{O}_\mathcal{S}$, any label $\mathcal{Q}_i$ such that $v \in \mathcal{Q}_i$ can be used (at least one such exists because otherwise $f(v) \in \cap_{j \in I_{\mathcal{O}_\mathcal{S}}} \mathcal{H}_j$, leading to the same contradiction).

Now for each $\alpha > 0$, $\alpha \in \mathbb{Z}$, define a triangulation $\mathbb{T}^\alpha$ of $\mathcal{O}_\mathcal{S}$ such that each simplex of $\mathbb{T}^\alpha$ has diameter $\frac{1}{\alpha}$. Apply Sperner's lemma for each $\mathbb{T}^\alpha$ to obtain a distinguished simplex $\mathrm{co}\{v_1^\alpha, \ldots, v_{\kappa+1}^\alpha\}$ and its barycenter x^α. The set $\{x^\alpha\}$ defines a bounded sequence in $\mathcal{O}_\mathcal{S}$ which has a convergent subsequence, which by abuse of notation is again denoted $\{x^\alpha\}$. We have $\lim_{\alpha \to \infty} x^\alpha = \overline{x} \in \mathcal{O}_\mathcal{S}$, since $\mathcal{O}_\mathcal{S}$ is closed. Also, by construction $v_i^\alpha \to \overline{x}$, $i \in I_{\mathcal{O}_\mathcal{S}}$. By Sperner's lemma we know that $v_i^\alpha \in \mathcal{Q}_i$, $i \in I_{\mathcal{O}_\mathcal{S}}$, so by continuity of $f(x)$ this implies $\overline{x} \in \overline{\mathcal{Q}}_i$, $i \in I_{\mathcal{O}_\mathcal{S}}$. That is, $\cap_{i \in I_{\mathcal{O}_\mathcal{S}}} \overline{\mathcal{Q}}_i \neq \emptyset$.

We conclude there exists a point $\overline{x} \in \mathcal{O}_\mathcal{S}$ such that $\overline{x} \in \cap_{i \in I_{\mathcal{O}_\mathcal{S}}} \overline{\mathcal{Q}}_i$. We claim that if $\overline{x} \in \overline{\mathcal{Q}}_i$, then $f(\overline{x}) \notin \mathrm{int}(\mathcal{H}_i)$. First, if $\overline{x} \in \mathcal{Q}_i$, then by definition, $f(\overline{x}) \in \mathcal{H}_i^c$ so $f(\overline{x}) \notin \mathrm{int}(\mathcal{H}_i)$. Second, if $\overline{x} \in \partial\mathcal{Q}_i$, one can show by continuity of f that $f(\overline{x}) \in \partial\mathcal{H}_i$, so again $f(\overline{x}) \notin \mathrm{int}(\mathcal{H}_i)$. We conclude $f(\overline{x}) \notin \mathrm{int}(\mathcal{H}_i)$, $i \in I_{\mathcal{O}_\mathcal{S}}$. This contradicts that $\mathcal{H}^\star = \mathbb{S}^{m-1}$. □

7 Discussion

In this section we compare our result to existing results in the literature. First, this paper only regards the case when $m > 1$, but we can easily recover the result for $m = 1$ found in Theorem 1 of [30]. When $m = 1$, the problem of a topological obstruction becomes a problem of finding a continuous function $f : \mathcal{O}_\mathcal{S} \to \mathbb{S}^0$ satisfying $f(x) \in \mathcal{C}(x)$, $x \in \mathcal{O}_\mathcal{S}$. As $\mathbb{S}^0$ consists of only two points, this means that such a continuous f can only be a constant function. Thus, a necessary and sufficient condition for finding the required f is that all $\mathcal{C}(x)$, $x \in \mathcal{O}_\mathcal{S}$, contain the same point. If we define $\mathcal{C}(\mathcal{O}_\mathcal{S}) := \cap_{x \in \mathcal{O}_\mathcal{S}} \mathcal{C}(x)$, then this is equivalent to saying $\mathcal{B} \cap \mathcal{C}(\mathcal{O}_\mathcal{S}) \neq \emptyset$. This is precisely the necessary and sufficient condition given in [30].

The case when the state space has been triangulated so that any intersection of $\mathcal{O}$ with $\mathcal{S}$ is along a face of $\mathcal{S}$ not containing v_0 has been extensively studied; see [6,2,7]. The key difference with the present case is that when $\mathcal{O}_{\mathcal{S}}$ is a face of $\mathcal{S}$, then $\mathrm{int}(\mathcal{O}_{\mathcal{S}}) \subset \partial\mathcal{S}$, so there are constraints on f in the relative interior of $\mathcal{O}_{\mathcal{S}}$. In the present case, $\mathrm{int}(\mathcal{O}_{\mathcal{S}}) \subset \mathrm{int}(\mathcal{S})$, so f has no constraints in the relative interior of $\mathcal{O}_{\mathcal{S}}$. For the sake of the present argument, suppose that $m = \kappa < n-1$ (the general case can also be handled). Following [6], suppose that $\mathcal{O}_{\mathcal{S}} = \mathrm{co}\{v_1, \ldots, v_{m+1}\}$. Then $\mathcal{O}_{\mathcal{S}} \subset \mathcal{F}_j$, $j = m+2, \ldots, n$, and for all $x \in \mathcal{O}_{\mathcal{S}}$, $\mathcal{C}(x) \subset \mathcal{C}_j$, $j = m+2, \ldots, n$. Hence, $\mathcal{H}_k \subset \mathcal{C}_j$, $k \in I_{\mathcal{O}_{\mathcal{S}}}$, $j = m+2, \ldots, n$, so $\mathcal{H} \subset \mathcal{C}_j$, $j = m+2, \ldots, n$. Now if we assume $\mathcal{H} = \mathbb{S}^{m-1}$ yet $\mathcal{H} \subset \mathcal{C}_j$, $j = m+2, \ldots, n$, it means that $\mathcal{B}$ is parallel to $\mathcal{F}_j$, $j = m+2, \ldots, n$. This is the essence of Proposition 8.2 and Remark 8.2 of [6] saying that with $v_0 = 0$, $\mathcal{B} \subset \mathrm{sp}\{v_1, \ldots, v_{m+1}\}$. It is shown in Theorem 7.3 of [6] that when $\mathcal{B} \cap \mathcal{C}(\mathcal{O}_{\mathcal{S}}) = \mathbf{0}$, with $\mathcal{C}(\mathcal{O}_{\mathcal{S}})$ as above, then there exists a closed-loop equilibrium in $\mathcal{O}_{\mathcal{S}}$ using any continuous state feedback satisfying the invariance conditions. We recover this finding here via the following result that relates the statement that $\mathcal{H} = \mathbb{S}^{m-1}$ to the condition on $\mathcal{B} \cap \mathcal{C}(\mathcal{O}_{\mathcal{S}})$.

Lemma 10 *Suppose* $v_0 = 0$*,* $\mathcal{B} = \mathrm{sp}\{v_1, \ldots, v_{m+1}\}$*, and* $\mathcal{O}_{\mathcal{S}} = \mathrm{co}\{v_1, \ldots, v_{m+1}\}$*. If* $\mathcal{B} \cap \mathcal{C}(\mathcal{O}_{\mathcal{S}}) = \mathbf{0}$*, then* $\mathcal{H} = \mathbb{S}^{m-1}$*.*

Proof Using the definition of the matrix M from Section 3, we note that $\mathcal{B} = \mathrm{sp}\{v_1, \ldots, v_{m+1}\}$ implies that, with $\tilde{h}_i = M^T h_i$ and dropping the tilde's, we have

$$h_{m+2}, \ldots, h_n = 0 \quad h_1, \ldots, h_{m+1} \neq 0\,. \tag{11}$$

By our index convention, the i-th facet of $\mathcal{O}_{\mathcal{S}}$ is $\mathcal{F}_i^{\mathcal{O}} = \{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{m+1}\}$. Thus, for $x \in \mathrm{int}(\mathcal{F}_i^{\mathcal{O}})$, $I(x) = \{1, 2, ..., i-1, i+1, ..., m+1\}$. By (4), $\mathcal{C}(x) = \{y \in \mathbb{S}^{m-1} \mid h_j \cdot y \leq 0, j = i, m+2, \ldots, n\}$. By (11), $\mathcal{C}(x) = \mathcal{C}_i$, so by definition $\mathcal{H}_i = \mathcal{C}_i$, which is a half-sphere. After the transformation through M, the assumption $\mathcal{B} \cap \mathcal{C}(\mathcal{O}_{\mathcal{S}}) = \mathbf{0}$ becomes $\cap_{i=1}^{m+1} \mathcal{C}_i = \emptyset$.

Suppose by way of contradiction that $\mathcal{H} \neq \mathbb{S}^{m-1}$. Consider the complements $\mathcal{H}_i^c = \mathbb{S}^{m-1} \setminus \mathcal{H}_i$. Since $\mathcal{H} \neq \mathbb{S}^{m-1}$, and $\mathcal{H}_i = \mathcal{C}_i$, there exists $x \in \mathbb{S}^{m-1}$ such that

$$h_i \cdot x > 0\,, i = 1, \ldots, m+1\,. \tag{12}$$

Now consider $-x \in \mathbb{S}^{m-1}$. By (12), $-x \in \mathcal{C}_i$ for all $i = 1, \ldots, m+1$, a contradiction. □

The subject of topological obstructions in the RCP has also been directly addressed in several recent papers. The approach of this paper is generally significantly deeper than the previous work. We comment on the relationships and differences between this paper and existing results.

In [24], the dimension of $\mathcal{B}$ is limited to $m = 2$. However, there are no strong assumptions on the geometric structure and dimension of $\mathcal{O}_{\mathcal{S}}$. A similar strategy as the one in this paper is used to iteratively build up a map on the skeleton of $\mathcal{O}_{\mathcal{S}}$ to a one-dimensional sphere. In the case of $m = 2$, all maps $f : \mathbb{S}^k \to \mathbb{S}^1$, $k > 1$, are automatically null-homotopic. That is not the case for maps $f : \mathbb{S}^k \to \mathbb{S}^{m-1}$, $m > 2$ [17]. Thus, extending boundary maps presents a key issue dealt with in this paper, and we do so using the theory of absolute retracts. The final result of [24] is that a solution to the problem of a topological obstruction in the case of $m = 2$ exists if and only if any valid boundary function is null-homotopic. In this paper, we give a solvability condition in terms of cones that relate $\mathcal{B}$ to $\mathcal{O}_{\mathcal{S}}$. These conditions arise more directly from the problem data. Additionally, we can recover the result for $m = 2$ as follows. Assuming that $\mathcal{H}_j \neq \mathbb{S}^{m-1}$ for all $j = 1, \ldots, m$, every boundary function satisfying the invariance conditions will have its image in $\mathcal{H} = \cup_j \mathcal{H}_j$. Hence, if $\mathcal{H} \neq \mathbb{S}^{m-1}$, which we proved to be a sufficient and necessary [1] condition for the existence of the solution to the problem of a topological obstruction, then any valid boundary function will be non-surjective. Thus, it will be null-homotopic by Lemma 2.

The paper [25] (supplemented by [23]) deals with the case of $n = 2$ and $n = 3$, and the central case of interest is when $\dim \mathcal{O}_{\mathcal{S}} = m = 2$. The results of that paper do not require the assumption that

[1] Up to a degenerate case where $\mathcal{H} = \mathbb{S}^{m-1}$, but $\mathcal{H}^* \neq \mathbb{S}^{m-1}$.

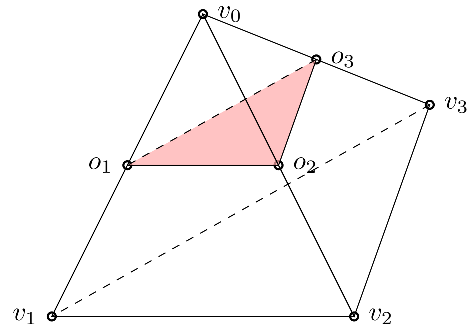

Fig. 4 One of the configurations of set $\mathcal{O}_\mathcal{S}$ for $n = 3$ and $\dim \mathcal{O}_\mathcal{S} = 2$ discussed in [25].

$\mathcal{O}_\mathcal{S} \cap \mathcal{F}_0 = \emptyset$, as we do. In the context of this paper, this would mean that $\mathcal{H}_i \subset \mathbb{S}^1$ only exist for $i = 1, 2, 3$, and it is possible that some of those $\mathcal{H}_i$'s in fact equal the entire sphere. Because of the low dimensions of objects involved, a case by case investigation using retraction theory and Sperner's lemma is performed. Thus, while the results of [25] can be subsumed into the final results of this paper, it is possible to employ a much less sophisticated strategy to obtain them in a much simpler form. For instance, one of the situations described in [25] is when $\mathcal{O}_\mathcal{S}$ is a triangle with $I(o_i) = \{0, i\}$ for $i = 1, 2, 3$. This situation is presented in Figure 4 (modified from [25]). Clearly, in that case, $\mathcal{H}_i = \mathcal{C}_i$, and, assuming the non-degeneracy condition (A3), each $\mathcal{H}_i = \mathcal{C}_i \subset \mathbb{S}^1$ is a half-sphere. It is easy to show that the union of these $\mathcal{H}_i$'s, or their interiors, will not cover the whole sphere if and only if all three of them intersect at some point, i.e., if $\mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{C}_3 \neq \emptyset$. Thus, using different notation, the results of this paper recover the result of [25], which states that there exists a solution to the problem of a topological obstruction if and only if $\mathcal{B} \cap \mathcal{C}(\mathcal{O}_\mathcal{S}) \neq \mathbf{0}$.

The paper [8] uses a purely algebraic strategy to solve the problem of a topological obstruction in a case where the position of $\mathcal{O}_\mathcal{S}$ with respect to $\mathcal{S}$ is very constrained, i.e., $\mathcal{O}_\mathcal{S}$ is positioned symmetrically within $\mathcal{S}$ in some sense. There also exist additional constraints on the position of $\mathcal{B}$ with respect to $\mathcal{S}$. This paper uses Sperner's lemma, also used by our paper, to again show that $\mathcal{B} \cap \mathcal{C}(\mathcal{O}_\mathcal{S}) \neq \mathbf{0}$ is a sufficient and necessary condition for the existence of a solution to the problem of a topological obstruction. However, the methods used in that paper are not topological, instead they depend on the position of $\mathcal{O}_\mathcal{S}$.

Finally, [20] deals with a similar problem to the one of a topological obstruction, where all the functions involved are required to be affine. This paper is of interest as the final result given is very similar in flavour to the one presented in our paper, i.e., the problem of affine obstructions is solvable if and only if the union of some cones does not cover the whole space. As this has been obtained by entirely different methods from our paper, this leads to the question of the relationship and gap (or lack thereof) between the problems of an affine obstruction and a topological obstruction. It has been conjectured in [24] that there is in fact no such gap.

8 Conclusion

This paper provides a complete characterization of the topological obstruction to solving the reach control problem by continuous state feedback. We apply both algebraic topology and Sperner's lemma. We showed that the existence of a topological obstruction depends on whether the union of certain cones associated with $\mathcal{O}_\mathcal{S}$, the set of possible equilibria in the simplex, covers a sphere $\mathbb{S}^{m-1}$. As such, our results are more closely tied to the problem data, in contrast with prior work on the problem. Up to a degenerate case, the cone condition forms a single sufficient and necessary condition for the existence of a solution to the problem of a topological obstruction. Topics for future work include removing the technical gap between the obtained sufficient and necessary conditions, as well as relaxing the assumptions present in the paper; in particular, removing the assumption that $\mathcal{O}_{\mathcal{S}}$ is a simplex. As the fundamental issue in this paper concerns the way the constraints posed on facets of $\mathcal{O}_{\mathcal{S}}$ interact, this gives rise to a possible nerve-theoretical approach to solving the problem.

References

1. J. Andres. Topological principles for ordinary differential equations. *Handbook of Differential Equations: Ordinary Differential Equations*. A. Cañada, P. Drábek, A. Fonda, eds. Vol. 3, pp. 1–101, 2006.
2. G. Ashford and M. E. Broucke. Time-varying affine feedback for each control on simplices. *Automatica*, vol. 49, issue 5, pp. 1365–1369, 2013.
3. K. C. Border. *Fixed point theorems with applications to economics and game theory*. Cambridge University Press, 1989.
4. K. Borsuk. *Theory of retracts*. Państwowe Wydawnictwo Naukowe, 1967.
5. G. E. Bredon. *Topology and Geometry*. Springer, 1997.
6. M. E. Broucke. Reach control on simplices by continuous state feedback. *SIAM Journal on Control and Optimization*. Vol. 48, no. 5, pp. 3482–3500, 2010.
7. M. E. Broucke and M. Ganness. Reach control on simplices by piecewise affine feedback. *SIAM Journal on Control and Optimization*. Vol. 52, no. 5, pp. 3261–3286, 2014.
8. M. E. Broucke, M. Ornik, and A. Mansouri. A topological obstruction in a control problem. *Systems and Control Letters*. Submitted August 2015.
9. L. Danzer, B. Grünbaum, and V. Klee. Hellys theorem and its relatives. *Convexity*. V. Klee, ed., pp. 101-180, 1963.
10. J. F. Davis and P. Kirk. *Lecture Notes in Algebraic Topology*. American Mathematical Society, 2001.
11. T. tom Dieck. *Algebraic Topology*. European Mathematical Society, 2008.
12. S. Eilenberg. Cohomology and continuous mappings. *The Annals of Mathematics*. Vol. 41, no. 1, pp. 231–251, 1940.
13. K. Elbassioni and H. R. Tiwary. On a cone covering problem. *Computational Geometry*. Vol. 44, no. 3, pp. 129–134, 2011.
14. L. C. G. J. M. Habets, P. J. Collins, and J. H. van Schuppen. Reachability and control synthesis for piecewise-affine hybrid systems on simplices. *IEEE Transactions on Automatic Control*. no. 51, pp. 938–948, 2006.
15. L. C. G. J. M. Habets and J. H. van Schuppen. A control problem for affine dynamical systems on a full-dimensional polytope. *Automatica*. no. 40, pp. 21–35, 2004.
16. M. K. Helwa and M. E. Broucke. Reach control of single-input systems on simplices using multi-affine feedback. *Mathematical Theory of Networks and Systems*. pp. 748–755, 2014.
17. H. Hopf. Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche. *Mathematische Annalen*. Vol. 104, no. 1, pp. 637–665, 1931.
18. H. Li, D. Hestenes, and A. Rockwood, "Spherical conformal geometry with geometric algebra", in *Geometric Computing with Clifford Algebras: Theoretical Foundations and Applications in Computer Vision and Robotics*, G. Sommer, ed., pp. 61–75, 2001.
19. K. Mehta. *A Topological Obstruction in a Control Problem*. M.A.Sc. Thesis, Department of Electrical and Computer Engineering. University of Toronto, 2012.
20. M. Moarref, M. Ornik, and M. E. Broucke. An obstruction to solvability of the reach control problem using affine feedback. *Automatica*. Vol. 71, pp. 229-236, 2016.
21. M. Müger, Supplements on geometric topology. *A Taste of Topology*. 2009.
22. G. L. Naber. *Topological Methods in Euclidean Spaces*. Dover, 2000.
23. M. Ornik and M. E. Broucke, Some results on an affine obstruction to reach control. *arXiv:1507.02957 [math.OC]*, 2015.
24. M. Ornik and M. E. Broucke. A topological obstruction to reach control by continuous state feedback. *IEEE Conference on Decision and Control*. pp. 2258–2263, 2015.
25. M. Ornik and M. E. Broucke. On a Topological Obstruction in the Reach Control Problem. *Mathematical and Computational Approaches in Advancing Modern Science and Engineering*. Proc. of AMMCS-CAIMS 2015, Springer, 2016.
26. T. Przymusiński. Collectionwise normality and absolute retracts. *Fundamenta Mathematicae*. Vol. 98, no. 1, pp. 61-73, 1978.
27. D. Repovs and P. V. Semenov. *Continuous Selections of Multivalued Mappings*. Springer, 2013.
28. C. V. Robinson. Spherical theorems of Helly type and congruence indices of spherical caps. *American Journal of Mathematics*. Vol. 64, no. 1, pp. 260–272, 1942.
29. B. Roszak and M. E. Broucke. Necessary and sufficient conditions for reachability on a simplex. *Automatica*. Vol. 42, no. 11, pp. 1913–1918, 2006.
30. E. Semsar-Kazerooni and M. E. Broucke. Reach controllability of single input affine systems. *IEEE Transactions on Automatic Control*. Vol. 59, no. 3, pp. 738–744, 2014.

Appendix

Lemma 11 ([18]) *Suppose (A3) holds. Let $y \in \mathbb{S}^{m-1}$. Suppose $y \in \mathcal{C}_j$. The stereographic projection of $\mathcal{C}_j \backslash \{-y\}$ centered at $-y$ equals:*

(i) a closed half-space in $\mathbb{R}^{m-1}$ if $h_j \cdot y = 0$,
(ii) an $m-1$-dimensional closed ball in $\mathbb{R}^{m-1}$ if $h_j \cdot y < 0$.

Lemma 12 *Suppose (A3) and (A4) hold. $\mathcal{H}$ is locally contractible.*

Proof Since local contractibility is a local property, we need only study a neighborhood of any point in $\mathcal{H}$. To that end, let $x \in \mathcal{H}$ and suppose without loss of generality $x \in \mathcal{H}_1 \cap \cdots \cap \mathcal{H}_r$, and $x \notin \mathcal{H}_{r+1}, \ldots, \mathcal{H}_{\kappa+1}$. Since all $\mathcal{H}_j$'s are closed in $\mathbb{S}^{m-1}$ there is a neighborhood $\mathcal{W}$ of x such that $\mathcal{W} \cap \mathcal{H} = \mathcal{W} \cap \bigcup_{j=1}^{r} \mathcal{H}_j$. Now consider $-x$. It is certainly outside some neighborhood of x. We will shrink $\mathcal{W}$ so that $-x \notin \mathcal{W}$. We will prove that $\mathcal{T} = \left(\bigcup_{j=1}^{r} \mathcal{H}_j\right) \backslash \{-x\}$ is locally contractible, from which it follows $\mathcal{W} \cap \mathcal{H}$ is locally contractible.

We use a stereographic projection centered at $-x$ of $\mathbb{S}^{m-1} \backslash \{-x\}$ into $\mathbb{R}^{m-1}$. By Lemma 11, this projection homeomorphically maps $\mathcal{C}_i \backslash \{-x\}$ to either a closed half-space in $\mathbb{R}^{m-1}$ (if $h_i \cdot x = 0$), or to a closed ball in $\mathbb{R}^{m-1}$, if $h_i \cdot x < 0$. Since $\mathcal{H}_j = \mathcal{C}(x)$ for $x \in \mathrm{int}(\mathcal{F}_j^{\mathcal{O}})$, $j \in I_{\mathcal{O}_\mathcal{S}}$, each $\mathcal{H}_j \backslash \{-x\}$ is the intersection of sets $\mathcal{C}_i \backslash \{-x\}$, so $\mathcal{T}$ is the union of intersections of sets $\mathcal{C}_i \backslash \{-x\}$. By Lemma 11, each $\mathcal{C}_i \backslash \{-x\}$ is mapped by the same homeomorphism into a convex set: either a half-space or a closed ball. Thus, each $\mathcal{H}_j \backslash \{-x\}$ is mapped into a convex set. Finally, $\mathcal{T}$ is homeomorphically deformed into a finite union of convex sets. By Lemma 1, it is locally contractible. □

Lemma 13 *Suppose (A3) and (A4) hold. Also suppose $\mathcal{H} \neq \mathbb{S}^{m-1}$ and $\bigcap_{\mathcal{H}_j \neq \mathbb{S}^{m-1}} \mathcal{H}_j \neq \emptyset$. Then $\mathcal{H}$ is contractible.*

Proof Let $\mathcal{Y} := \cap_{\mathcal{H}_j \neq \mathbb{S}^{m-1}} \mathcal{H}_j$. Since each $\mathcal{H}_j$ is itself an intersection of $\mathcal{C}_j$'s, $\mathcal{Y}$ satisfies Lemma 5. Let $I' \subset I$ be the index set of $\mathcal{C}_j$'s whose intersection forms $\mathcal{Y}$. By Lemma 5 there exists $x \in \mathcal{Y} \subseteq \mathcal{H}$ such that $h_k \cdot x < 0$ for all $k \in I'$. Since $h_j \cdot (-x) > 0$ for all $k \in I'$, we know $-x \notin \mathcal{H}_j$ for any $\mathcal{H}_j \subset \mathcal{H}$. Thus, $-x \notin \mathcal{H}$. Consider geodesics on $\mathbb{S}^{m-1}$ coming out of x. Because the antipodal point $-x$ is not in $\mathcal{H}$, there exists a unique geodesic $f_{x'}$ between x and any point $x' \in \mathcal{H}_j$ for any $\mathcal{H}_j \subset \mathcal{H}$. Since each $\mathcal{H}_j$ is Robinson-convex (see [28,9]), the entire path of geodesic $f_{x'}$ lies inside some $\mathcal{H}_j \subseteq \mathcal{H}$, as both x and x' are in $\mathcal{H}_j$. Thus, $\mathcal{H}$ is a star-shaped set with respect to geodesics on a sphere. By a repetition of the standard proof for star-shaped sets in Euclidean spaces, $\mathcal{H}$ is contractible. □