A topological obstruction to reach control by continuous state feedback

Melkior Ornik[1] and Mireille E. Broucke[1]

***Abstract*— The reach control problem (RCP) deals with finding a feedback control which drives the states of an affine control system to leave a simplex through a predetermined facet. In analogy with Brockett's obstruction to continuous feedback stabilization, a topological obstruction to solving the RCP by continuous feedback results in a deep necessary condition for solvability. In this paper, we interpret the problem of topological obstruction as a problem of existence of null-homotopic maps on spheres. This results in a complete and easily implementable characterization of the topological obstruction for the case of systems with two inputs.**

I. Introduction

We study the Reach Control Problem (RCP) for affine systems defined on simplices. The problem is to drive the states of an affine system defined on a simplex to reach and exit a prescribed facet of the simplex in finite time without first leaving the simplex [13], [22]. The RCP is motivated by problems of hybrid systems, particularly control problems involving safety and reachability specifications [25]; it is a natural building block in such problems.

An increasing number of applications have recently been identified to be amenable to an RCP approach. These include aircraft control [5], biomolecular networks [6], material transport [2], aggressive maneuvers of mechanical systems [24], automated anesthesia [11], robotic manipulators [19], mobile robots in complex environments [4] and process control [15]. On the theoretical side, most research has focused on identifying classes of feedbacks to solve the RCP, including affine state feedback [13], [22], continuous state feedback [9], time-varying affine feedback [1], piecewise affine feedback [10] and multi-affine feedback [16]. Another aspect of the problem which has been explored is the choice of triangulation of the state space. A special triangulation that allows closed-loop equilibria only to appear on faces of simplices has enabled a more structural, geometric analysis of the problem [9], [1], [10]. An alternative triangulation has lead to further discoveries about the problem [23], [16].

In this paper we focus on more general triangulations (as do [23], [16]), and we focus on continuous state feedback. In parallel with [21], we announce the existence of a topological obstruction to solving the RCP by continuous state feedback. We provide a brief history on this aspect of the problem.

The possibility of a topological obstruction to solving the RCP by continuous state feedback first arose in a disguised form in Theorem 8.3 of [9], where it was observed that for the special triangulation and under certain conditions on the system, equilibria always appeared in the simplex using continuous state feedback. However, at that stage the phenomenon was not understood in topological terms. Further efforts to characterize the appearance of equilibria appeared in [10] where so-called reach control indices were introduced to catalogue the placement of equilibria under the special triangulation. Again, the topological nature of the problem had not yet been grasped.

The contribution of this paper is to remove the assumption of a special triangulation as well as to introduce topological tools to tackle the problem. This research is most closely related to [21]. The differences between this paper and [21] are as follows. [21] provides results solely for two- and three-dimensional simplices, but without limitations on the number of inputs. On the other hand, the dimension of $\mathcal{S}$ in this paper is not limited, but the system is required to have two inputs. Also, the approaches are entirely different: [21] is based on retraction theory, whereas this paper uses the theory of null-homotopic functions. As such, the conditions for a topological obstruction are also quite different. Further research is needed to relate the two approaches.

Notation: Notation $\mathbb{B}^n$ denotes the closed unit ball in $\mathbb{R}^n$ centered at the origin and $\mathbb{S}^{n-1} = \partial\mathbb{B}^n$ denotes the unit sphere in $\mathbb{R}^n$. If $F : \mathcal{X} \to \mathcal{Y}$ is a map whose domain $\mathcal{X}$ is a closed set, then the notation $\partial F : \partial\mathcal{X} \to \mathcal{Y}$ refers to the *boundary map* which coincides with F on the boundary of $\mathcal{X}$. If $A \subset \mathbb{R}^n$, A° denotes its (relative) interior.

II. Reach Control Problem

We consider an $n-$dimensional simplex $\mathcal{S} := \text{co}\{v_0, v_1, \cdots, v_n\}$ with vertex set $V_{\mathcal{S}} := \{v_0, v_1, \cdots, v_n\}$ and facets $\mathcal{F}_0, \cdots, \mathcal{F}_n$ (the facet is indexed by the vertex it does not contain). Let h_i, $i \in \{0, \cdots, n\}$, be the unit normal vector to the facet $\mathcal{F}_i$ pointing outside the simplex, and let $\mathcal{F}_0$ be the *exit facet*. Define $I := \{1, \cdots, n\}$, and for $x \in \mathcal{S}$, let $I(x)$ be the minimal index set such that $x \in \text{co}\{v_i \mid i \in I(x)\}$. We also define $J(x) \subset I$ to be the set of indices of facets containing x, given by $J(x) = \{j \in I \mid x \in \mathcal{F}_j\}$. Note that $I(x) = I \setminus J(x)$.

Consider the affine control system defined on $\mathcal{S}$:

$$\dot{x} = Ax + Bu + a\,, \qquad x \in \mathcal{S}\,, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $a \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times m}$, and $\text{rank}(B) = m$. Let $\mathcal{B} = \text{Im}(B)$, the image of B. Also, let $\phi_u(t, x_0)$ denote the trajectory of (1) under a control law u starting from $x_0 \in \mathcal{S}$.

Problem 1: (Reach Control Problem (RCP)) Consider system (1) defined on a simplex $\mathcal{S}$. Find a continuous state

* This work is supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

[1] Department of Electrical & Computer Engineering, University of Toronto, Ontario, Canada.

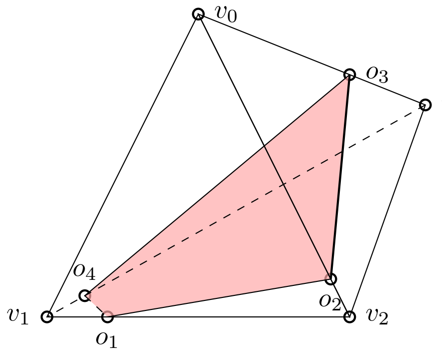

Fig. 1. 3-dimensional simplex $\mathcal{S}$ with a possible set of potential equilibria $\mathcal{O}_{\mathcal{S}}$.

feedback $u(x)$ such that for every $x_0 \in \mathcal{S}$, there exists $T \geq 0$ such that

(i) $\phi_u(t, x_0) \in \mathcal{S}$ for all $0 \leq t \in [0, T]$.
(ii) $\phi_u(T, x_0) \in \mathcal{F}_0$.
(iii) $\phi_u(T + \varepsilon, x_0) \notin \mathcal{S}$ for all sufficiently small $\varepsilon > 0$.

Essentially, the RCP requires that closed-loop trajectories leave $\mathcal{S}$ in finite time through the exit facet $\mathcal{F}_0$.

It is clear that a necessary condition for $u(x)$ to solve the RCP is that there are no closed-loop equilibria in $\mathcal{S}$, i.e. $Ax + Bu(x) + a \neq 0$, $x \in \mathcal{S}$. Closed-loop equilibria of (1) can only appear in the affine space $\mathcal{O} := \{x \in \mathbb{R}^n | Ax + a \in \mathcal{B}\}$ [22], [9]. Thus, we are interested in equilibria in the set

$$\mathcal{O}_{\mathcal{S}} := \mathcal{S} \cap \mathcal{O}.$$

Since $\mathcal{O}$ is an affine space, either $\mathcal{O}_{\mathcal{S}} = \emptyset$ or $\mathcal{O}_{\mathcal{S}}$ is a convex polytope in $\mathcal{S}$ with a dimension $0 \leq \kappa \leq n$. Let $V_{\mathcal{O}_{\mathcal{S}}} = \{o_1, \cdots, o_{k+1}\}$ denote the set of vertices of $\mathcal{O}_{\mathcal{S}}$. An example of $\mathcal{O}_{\mathcal{S}}$ is shown in Figure 1.

A second necessary condition for solving the RCP is that velocity vectors $Ax + Bu(x) + a$ must point inward at points in the facets $\mathcal{F}_i$, $i \in I$ [12]. To formalize this requirement, for $x \in \mathcal{S}$, define the closed, convex cone

$$\mathcal{C}(x) := \left\{ \; y \in \mathbb{R}^n \mid h_j \cdot y \leq 0, \;\; j \in J(x) \; \right\}.$$

Figure 2 illustrates the cones $\mathcal{C}(x)$ as shaded cones attached at various $x \in \mathcal{S}$ since they are used to characterize tangent velocity vectors. Notice that for $x \in \mathcal{S} \setminus \mathcal{F}_0$, $\mathcal{C}(x)$ is exactly the Bouligand tangent cone to $\mathcal{S}$ at v_0, $T_{\mathcal{S}}(x)$. At $x \in \mathcal{F}_0$, $\mathcal{C}(x)$ includes directions pointing out of $\mathcal{S}$. The requirement that velocity vectors must point inwards can be formally stated as

$$Ax + Bu(x) + a \in \mathcal{C}(x), \quad x \in \mathcal{S}\,. \tag{2}$$

Let $u(x)$ be a continuous state feedback. Define $f(x) = Ax + Bu(x) + a$. Notice that, by the definition of $\mathcal{O}_{\mathcal{S}}$, $f(x) \in \mathcal{B}$ for $x \in \mathcal{O}_{\mathcal{S}}$. From the foregoing discussion, a necessary condition for $u(x)$ to solve the RCP is that there exists $f : \mathcal{O}_{\mathcal{S}} \to \mathcal{B} \setminus \{0\}$ such that $f(x) \in \mathcal{C}(x)$, $x \in \mathcal{O}_{\mathcal{S}}$. We now transform this statement so that explicit consideration of $\mathcal{B}$ can be avoided.

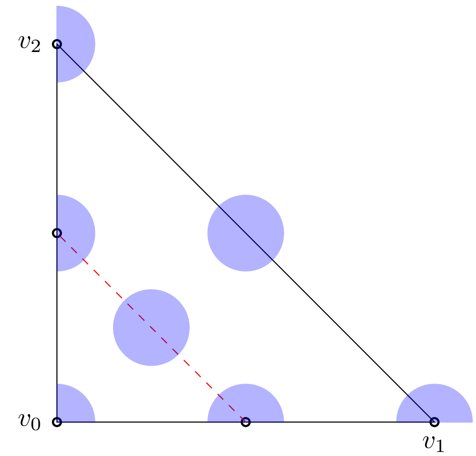

Fig. 2. Illustration of $\mathcal{C}(x)$, depicted as blue cones at several points $x \in \mathcal{S}$. An example of $\mathcal{O}_{\mathcal{S}}$ is depicted by a dashed line.

Since $\mathcal{B} \subset \mathbb{R}^n$ is an m-dimensional subspace, we can identify it with $\mathbb{R}^m$ through a linear transformation $M \in \mathbb{R}^{n \times m}$ whose columns form an orthonormal basis of $\mathcal{B}$, such that $M^T M = I$ and $MM^T|_{\mathcal{B}} \equiv id$. This interprets the function $f : \mathcal{O}_{\mathcal{S}} \to \mathcal{B} \backslash \{0\}$ as $\frac{M^T f}{\|M^T f\|} = \tilde{f} : \mathcal{O}_{\mathcal{S}} \to \mathbb{S}^{m-1}$. The requirement $f(x) \in \mathcal{C}(x)$ says $h_j \cdot f(x) \leq 0$, $j \in J(x)$. Since $MM^T|_{\mathcal{B}} \equiv id$, and $f(x) \in \mathcal{B}$ for $x \in \mathcal{O}_{\mathcal{S}}$, $h_j \cdot f(x) = h_j \cdot MM^T f(x)$. Let $\tilde{h}_j := M^T h_j \in \mathbb{R}^m$. Then $f(x) \in \mathcal{C}(x)$ is equivalent to $\tilde{h}_j \cdot \tilde{f}(x) \leq 0$, $j \in J(x)$. In what follows we abuse notation and remove the tilde's from the variables f and h_j. We arrive at the main problem studied in this paper.

Problem 2: Does there exist a continuous function $f : \mathcal{O}_{\mathcal{S}} \to S^{m-1}$ satisfying

$$f(x) \in \mathcal{C}(x)\,, \qquad x \in \mathcal{O}_{\mathcal{S}}\,. \tag{3}$$

III. Background

Problem 2 will be tackled using the tools of algebraic topology [14]. In this section we give the necessary background. Particularly, we identify Problem 2 as one regarding maps on spheres and more particularly, we apply ideas from extension theory - whether a function can be extended from the boundary of set to its interior.

We say two topological spaces $\mathcal{X}$ and $\mathcal{Y}$ are *homeomorphic* if there exists a continuous function $f : \mathcal{X} \to \mathcal{Y}$ which is bijective and has a continuous inverse. Such an f is called a *homeomorphism*.

Theorem 3 ([7]): Let $\mathcal{P}$ be a convex κ-dimensional polytope. $\mathcal{P}$ is homeomorphic to $\mathbb{B}^\kappa$, and $\partial\mathcal{P}$ is homeomorphic to $\mathbb{S}^{\kappa-1} = \partial\mathbb{B}^\kappa$.

Next, we require some background from homotopy theory. A *path* is defined as a continuous function $f : [0, 1] \to \mathcal{Y}$, with endpoints at $f(0)$ and $f(1)$. The *reverse path* of path f is denoted by $\bar{f}$ and is defined by $\bar{f}(t) = f(1 - t)$. A *concatenation* of two paths f and g with $f(1) = g(0)$ is denoted by fg and formally defined by $fg(t) = f(2t)$ for $t \leq 1/2$ and $fg(t) = g(2t - 1)$ for $t > 1/2$. A *loop* is a closed path, i.e., a path f with $f(0) = f(1)$. We say $f(0)$ is the *basepoint* of the loop f.

Definition 4: Let $f, g : \mathcal{X} \to \mathcal{Y}$ be continuous maps. We say f is *homotopic* to g, denoted by $f \simeq g$, if there exists a continuous function $F : \mathcal{X} \times [0, 1] \to \mathcal{Y}$ such that $F(\cdot, 0) \equiv f$ and $F(\cdot, 1) \equiv g$.

For paths, we have the following notion.

Definition 5: Let $f, g : [0, 1] \to \mathcal{Y}$ be paths with the same endpoints, i.e. $f(0) = g(0)$ and $f(1) = g(1)$. We say f is *homotopic* to g, denoted by $f \simeq g$, if there exists a continuous function $F : [0, 1] \times [0, 1] \to \mathcal{Y}$ such that $F(\cdot, 0) \equiv f$, $F(\cdot, 1) \equiv g$, $f(0, \cdot) \equiv f(0)$, $f(1, \cdot) \equiv g(0)$.

Finally, we draw frow two other areas of algebraic topology: the extension problem and maps on spheres. The *extension problem* regards the following question: given a continuous map $f : \partial\mathcal{X} \to \mathcal{Y}$ defined on the boundary of a space $\mathcal{X}$, we would like to know if there exists a continuous extension of $F : \mathcal{X} \to \mathcal{Y}$ such that $F|_{\partial\mathcal{X}} \equiv f$. The terminology *topological obstruction* particularly refers to an obstruction to extending a continuous map. Of particular relevance to the extension problem is the notion of null-homotopic maps. We make use of several results classifying maps that are null-homotopic.

Definition 6: A continuous map $f : \mathcal{X} \to \mathcal{Y}$ is said to be *inessential* or *null-homotopic* if f is homotopic to a constant map $c(x) = y_0$, a single point in $\mathcal{Y}$. Otherwise f is said to be *essential*.

Lemma 7: If $f : \mathcal{X} \to \mathbb{S}^n$ is a continuous map such that $f(\mathcal{X}) \neq \mathbb{S}^n$, that is, f is not surjective, then f is null-homotopic.

Lemma 8 ([20]): If $f : \mathcal{X} \to \mathcal{Y}$ is a continuous map and $\mathcal{Y}$ is contractible, then f is null-homotopic.

Lemma 9 ([20]): Every continuous map $f : \mathbb{S}^n \to \mathbb{S}^1$ with $n \geq 2$ is null-homotopic.

Lemma 10 ([18]): Let $f, g : [0, 1] \to Y$ be paths with the same endpoints. Then, f is homotopic to g if and only if $f\bar{g}$ is null-homotopic. In particular, $f\bar{f}$ is null-homotopic.

The main result relating null-homotopic maps and the extension problem is the following important Extension Theorem (see [20]).

Theorem 11: A continuous map $f : \mathbb{S}^n \to \mathcal{Y}$ is null-homotopic if and only if f extends to a map $F : \mathbb{B}^{n+1} \to \mathcal{Y}$.

Proof: Suppose $f : \mathbb{S}^n \to \mathcal{Y}$ is inessential and let $H : \mathbb{S}^n \times [0, 1] \to \mathcal{Y}$ be a homotopy between $f(x) = H(x, 0)$ and a constant map $c(x) = H(x, 1)$. Note that every point $y \in \mathbb{B}^{n+1} \backslash \{0\}$ can be written uniquely as tx, where $t \in (0, 1]$ and $x \in \mathbb{S}^n$. Define $F : \mathbb{B}^{n+1} \to \mathcal{Y}$ by $F(tx) = H(x, 1 - t)$ for $x \in \mathbb{S}^n$ and $t \in (0, 1]$. Also define $F(0) = c(0)$. It is easy to check that F is continuous.

Conversely, suppose that $f : \mathbb{S}^n \to \mathcal{Y}$ extends to a map $F : \mathbb{B}^{n+1} \to \mathcal{Y}$. Define $H(x, t) : \mathbb{S}^n \times [0, 1] \to \mathcal{Y}$ by $H(x, t) := F(tx)$. Then H is a homotopy between f and a constant map $c(x) = F(0)$. ■

IV. Main Results

The main idea of our approach to Problem 2 is as follows. First, a technical lemma, Lemma 12, gives a useful property about the index sets $J(x)$. Then we examine the two dimensional polytopes in $\partial\mathcal{O}_\mathcal{S}$. We assume that on those polytopes there exists a continuous function which satisfies the requirements of Problem 2. Thanks to the conditions (2), this function can be shown, via Lemma 8 or Lemma 9, to be null-homotopic. In Lemma 13 an induction argument on the dimension of the boundary polytopes in $\mathcal{O}_\mathcal{S}$ shows that the proposed null-homotopic map defined on the two dimensional boundary polytopes of $\mathcal{O}_\mathcal{S}$ can be continuously extended to all of $\mathcal{O}_\mathcal{S}$ and still satisfy the conditions (2).

Once we have identified the existence of a continuous map satisfying (2) on the two dimensional boundary polytopes of $\mathcal{O}_\mathcal{S}$ as turnkey to the solution, we then turn, in Proposition 18, to identifying verifiable conditions for existence of such a map. The argument relies on homotopy properties of certain loops, expressed in Lemmas 16 and 17. Combining Lemma 13 and Proposition 18, we obtain the main result, which is Theorem 19.

Lemma 12: Let $\dim(\mathcal{O}_\mathcal{S}) = \kappa \geq 2$ and $\mathcal{O}_\mathcal{S} \cap \mathcal{F}_0 = \emptyset$. Suppose $o_1, o_2 \in V_{\mathcal{O}_\mathcal{S}}$ are such that $\overline{o_1 o_2}$ is a one-dimensional edge of $\mathcal{O}_\mathcal{S}$. Then, $I(o_1) \cup I(o_2) \neq I$. Equivalently, $J(o_1) \cap J(o_2) \neq \emptyset$.

Proof: Suppose by way of contradiction that $I(o_1) \cup I(o_2) = I$. Let $x = \frac{1}{2}o_1 + \frac{1}{2}o_2$. Since $\dim(\mathcal{O}_\mathcal{S}) \geq 2$, $x \in \overline{o_1 o_2} \subset \partial\mathcal{O}_\mathcal{S}$. By Lemma 1 of [23], $\partial\mathcal{O}_\mathcal{S} \subset \partial\mathcal{S}$, so $x \in \partial\mathcal{S}$. Let $o_1 = \sum_{i \in I(o_1)} \alpha_i v_i$ and $o_2 = \sum_{i \in I(o_2)} \beta_i v_i$ for some $\alpha_i > 0$ and $\beta_i > 0$. Then $x = \frac{1}{2}\sum_{i \in I(o_1)} \alpha_i v_i + \frac{1}{2}\sum_{i \in I(o_2)} \beta_i v_i = \sum_{k \in I} \lambda_k v_k$, where $\lambda_k = \alpha_k/2$ if $k \in I(o_1) \setminus I(o_2)$, $\lambda_k = \beta_k/2$ if $k \in I(o_2) \setminus I(o_1)$, and $\lambda_k = (\alpha_k + \beta_k)/2$ if $k \in I(o_1) \cap I(o_2)$. Note that hence $\lambda_k > 0$, $k \in I$. Thus, $x \in \mathcal{S}^\circ \cup \mathcal{F}_0$ but $\mathcal{O}_\mathcal{S} \cap \mathcal{F}_0 = \emptyset$, so $x \in \mathcal{S}^\circ$, a contradiction. ■

For each k, $0 \leq k \leq \kappa$, let $\partial_k\mathcal{O}_\mathcal{S}$ be the *k-skeleton* of $\mathcal{O}_\mathcal{S}$, i.e., the union of all k-dimensional edges of $\mathcal{O}_\mathcal{S}$. Particularly, $\partial_0\mathcal{O}_\mathcal{S} = V_{\mathcal{O}_\mathcal{S}}$, and $\partial_\kappa\mathcal{O}_\mathcal{S} = \mathcal{O}_\mathcal{S}$.

The next result allows us to reduce the problem of finding the topological obstruction on $\mathcal{O}_\mathcal{S}$ with an arbitrary dimension to determining whether there exists an obstruction on the skeleton of two-dimensional edges of $\mathcal{O}_\mathcal{S}$. The argument rests on the fact that every continuous map $g : \mathbb{S}^k \to \mathbb{S}^1$, $k \geq 2$, is null-homotopic. Hopf showed in [17] that such a claim is not generally true for maps $g : \mathbb{S}^k \to \mathbb{S}^r$ with $r \geq 2$.

Lemma 13: Let $\dim(\mathcal{O}_\mathcal{S}) = \kappa \geq 2$ and $m = 2$. If there exists $\partial_2 f : \partial_2\mathcal{O}_\mathcal{S} \to \mathbb{S}^1$ satisfying $\partial_2 f(x) \in \mathcal{C}(x)$ for $x \in \partial_2\mathcal{O}_\mathcal{S}$, then $\partial_2 f$ can be extended to $f : \mathcal{O}_\mathcal{S} \to \mathbb{S}^1$ satisfying $f(x) \in \mathcal{C}(x)$, $x \in \mathcal{O}_\mathcal{S}$.

Proof: The proof is by induction on the dimension k. The base case when $k = 2$ is the assumption of the lemma statement. Next, if $2 \leq k \leq \kappa - 1$ and there exists $\partial_k f : \partial_k\mathcal{O}_\mathcal{S} \to \mathbb{S}^1$ satisfying (3), then we will show that there exists $\partial_{k+1} f : \partial_{k+1}\mathcal{O}_\mathcal{S} \to \mathbb{S}^1$ satisfying (3) and such that $\partial_{k+1} f|_{\partial_k\mathcal{O}_\mathcal{S}} \equiv \partial_k f$. Finally, the result follows by noting that $\partial_\kappa\mathcal{O}_\mathcal{S} = \mathcal{O}_\mathcal{S}$.

Let $2 \leq k \leq \kappa - 1$ and suppose there exists $\partial_k f : \partial_k\mathcal{O}_\mathcal{S} \to \mathbb{S}^1$ satisfying (3). Since $\mathcal{O}_\mathcal{S}$ is a convex polytope, $\partial_{k+1}\mathcal{O}_\mathcal{S}$ is comprised of $(k + 1)$-dimensional convex polytopes, and those polytopes intersect in polytopes of lower dimension.

Take any $(k+1)$-dimensional polytope $\mathcal{A}$ in $\partial_{k+1}\mathcal{O}_\mathcal{S}$. Let $\partial\mathcal{A} \subset \partial_k\mathcal{O}_\mathcal{S}$ be the relative boundary of $\mathcal{A}$, consisting of k-dimensional polytopes in $\partial_k\mathcal{O}_\mathcal{S}$. By Theorem 3, $\mathcal{A}$ is homeomorphic to $\mathbb{B}^{k+1}$, and the same homeomorphism sends $\partial\mathcal{A}$ to $\partial\mathbb{B}^{k+1} = \mathbb{S}^k$. Via this homeomorphism $\partial_k f|_{\partial\mathcal{A}} : \partial\mathcal{A} \to \mathbb{S}^1$ can be understood as $\partial_k f|_{\partial\mathcal{A}} : \mathbb{S}^k \to \mathbb{S}^1$. Let $\mathcal{Y} := \text{Im}(\partial_k f|_{\partial\mathcal{A}})$.

We consider two cases. First, suppose $\mathcal{Y} \neq \mathbb{S}^1$. Since $\partial_k f|_{\partial\mathcal{A}}$ is a continuous map and $\partial\mathcal{A}$ is connected, $\mathcal{Y}$ must be a circular arc. Hence, it is trivially homeomorphic to $[0,1]$. The interval $[0,1]$ and hence $\mathcal{Y}$ are contractible. By Lemma 8 we obtain that $\partial_k f|_{\partial\mathcal{A}} : \mathbb{S}^{k+1} \to \mathcal{Y}$ is null-homotopic. By Theorem 11, $\partial_k f|_{\partial\mathcal{A}}$ can be extended to $f_\mathcal{A} : \mathcal{A} \to \mathcal{Y}$. Second, suppose $\mathcal{Y} = \mathbb{S}^1$. By Lemma 9, $\partial_k f|_{\partial\mathcal{A}} : \mathbb{S}^k \to \mathbb{S}^1$ is null-homotopic. By Theorem 11, $\partial_k f|_{\partial\mathcal{A}}$ can be extended to $f_\mathcal{A} : \mathcal{A} \to \mathcal{Y}$. (It is worth noting that we cannot use Lemma 9 for the first case because we are specifically interested in extensions that map into $\text{Im}(\partial_k f|_{\partial\mathcal{A}})$ in order for the argument below to go through.)

Next we show that $f_\mathcal{A}$ satisfies (3). Consider any $x \in \mathcal{A}$. If $x \in \partial\mathcal{A}$, then $f_\mathcal{A}(x) = \partial_k f(x)$, and by assumption $\partial_k f$ satisfies (3). Instead suppose $x \in \mathcal{A}^\circ$. In the two cases above we have constructed an extension that maps to $\mathcal{Y} = \text{Im}(\partial_k f|_{\partial\mathcal{A}})$. Thus, there exists $y \in \partial\mathcal{A}$ such that $f_\mathcal{A}(x) = \partial_k f(y)$. By assumption, $\partial_k f(y) \in \mathcal{C}(y)$. On the other hand, $\mathcal{C}(y) \subset \mathcal{C}(x)$. To see that, consider the line going through y and x. Since $\mathcal{A}$ is a convex polytope, there exists $z \in \partial\mathcal{A}$ such that $x \in \overline{yz}$. Then x is a convex combination of y and z. Thus, $I(y) \cup I(z) \subset I(x)$. In particular, $J(x) \subset J(y)$. Using the definition of $\mathcal{C}(x)$, we get $f_\mathcal{A}(x) \in \mathcal{C}(y) \subset \mathcal{C}(x)$.

Finally, we want to show that different $f_\mathcal{A}$ can be "glued" together to obtain $\partial_{k+1} f$. Precisely, if $\mathcal{A}$ and $\mathcal{A}'$ are different $(k+1)$-dimensional faces of $\mathcal{O}_\mathcal{S}$ and $x \in \mathcal{A} \cap \mathcal{A}'$, then $f_\mathcal{A}(x) = f_{\mathcal{A}'}(x)$. This follows because the $(k+1)$-dimensional faces of $\mathcal{O}_\mathcal{S}$ intersect on k-dimensional faces. Thus, if $x \in \mathcal{A}\cap\mathcal{A}'$, then $x \in \partial_k\mathcal{O}_\mathcal{S}$, and $f_\mathcal{A}(x) = \partial_k f(x) = f_{\mathcal{A}'}(x)$. The result is that $\partial_{k+1} f(x) := f_\mathcal{A}(x)$ if $x \in \mathcal{A}$ defines a continuous map on $\partial_{k+1}\mathcal{O}_\mathcal{S} = \cup\mathcal{A}$. ■

Assumption 14: (i) $h_i \neq 0$ for all $i \in I$.
(ii) $h_i \neq \lambda h_j$ for all $i, j \in I$ and all $\lambda < 0$.
(iii) $\mathcal{O}_\mathcal{S} \cap \mathcal{F}_0 = \emptyset$.

Part (i) of the above assumption ensures that all sets $\{y \in \mathbb{S}^1 | h_i^T y \leq 0\}$ are semicircles. Part (ii) ensures that no two of these semicircles are diametrically opposite.

The requirements of Assumption 14 can be removed, but they significantly contribute to the elegance of the stated results. So, unless noted otherwise, we are now assuming that Assumption 14 is true.

Let $\mathcal{A}$ be a two-dimensional polytope in $\partial_2\mathcal{O}_\mathcal{S}$ such that $\mathcal{A} = \text{co}\{o_1, \cdots, o_r\}$, where we are assuming without loss of generality that o_i's are ordered counterclockwise. In other words, the edges of $\mathcal{A}$ are $\overline{o_1 o_2}$, $\overline{o_2 o_3}$, ..., $\overline{o_r o_1}$. Consider a continuous map $F : \mathcal{A} \to \mathbb{S}^1$ such that $F(x) \in \mathcal{C}(x)$, $x \in \mathcal{A}$. Define $f_i := F|_{\overline{o_i o_{i+1}}}$, taking $o_{r+1} \equiv o_1$. Also let $\tilde{f}_i$ be the path traversing the shorter circular arc between $F(o_i)$ and $F(o_{i+1})$. Since reparametrizations do not change the homotopy properties of paths, without loss of generality we assume that f_i and $\tilde{f}_i$ traverse the arc in $\mathbb{S}^1$ with uniform speed. Let $b_1, \cdots, b_r \in \mathbb{S}^1$ be any vectors such that $b_i \in \mathcal{C}(o_i)$, $i = 1, \cdots, r$. Such b_i's exist, as otherwise some $\mathcal{C}(o_i)$ would be empty and the problem would clearly not be solvable (see Proposition 18 and Theorem 19). Define g_i to be the shorter arc in $\mathbb{S}^1$ between b_i and b_{i+1}. Let m_i be the path from $F(o_i)$ to b_i through the shorter circular arc in $\mathbb{S}^1$. An illustrative example of the paths defined above is given in Figure 3.

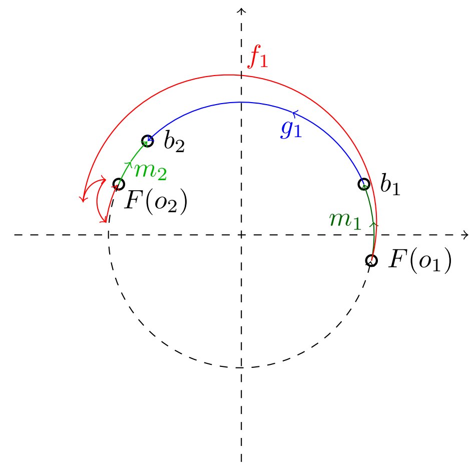

Fig. 3. An illustration of the different paths defined below. Paths m_1 and m_2 are the shortest paths from $F(o_1)$ to b_1 and $F(o_2)$ to b_2, respectively. Path g_1 is the shortest path from b_1 to b_2, and f_1 is a path (not necessarily the shortest) from $F(o_1)$ to $F(o_2)$. Path f_1 is drawn away from the circle for visibility. Path $\tilde{f}_1$ is not pictured, but is the shortest path from $F(o_1)$ to $F(o_2)$. In this case, $\tilde{f}_1$ equals $m_1 g_1 \overline{m}_2$.

Remark 15: In the remainder of this text, it will often be necessary to deal with the "shorter arc" between two points $b_i \in \mathcal{C}(o_i)$ and $b_{i+1} \in \mathcal{C}(o_{i+1})$. This presents an issue if both arcs between those points are of equal length π. Let those arcs be labeled $\mathcal{K}_1$ and $\mathcal{K}_2$. We will show that in that case, exactly one of the arcs $\mathcal{K}_i$ satisfies the following: for each point y on the arc, $h_j^T y \leq 0$ for all $j \in J(o_i) \cap J(o_{i+1})$.

Let us prove that. We note that by Assumption 14 and Lemma 12, there exists $k \in J(o_i) \cap J(o_{i+1})$. Now, $b_i \in \mathcal{C}(o_i)$ implies that $h_k^T b_i \leq 0$, and $h_k^T b_{i+1} \leq 0$. Since both arcs between b_i and b_{i+1} are of length π, this means that $b_i = -b_{i+1}$. So, $h_k^T b_i = h_k^T b_{i+1} = 0$. Thus, b_i and b_{i+1} are on the edges of the semicircle $\{y \in \mathbb{S}^1 | h_k^T y \leq 0\}$. This semicircle is hence exactly an arc between b_i and b_{i+1}. Hence, for each $k \in J(o_i) \cap J(o_{i+1})$, $\{y \in \mathbb{S}^1 | h_k^T y \leq 0\} = \mathcal{K}_1$ or $\{y \in \mathcal{S}^1 | h_k^T y \leq 0\} = \mathcal{K}_2$.

We note that $h_k^T b_i = h_k^T b_{i+1} = 0$ also implies that, for every $k \in J(o_i) \cap J(o_{i+1})$, h_k is perpendicular to b_i. As we are working with vectors in $\mathbb{R}^2$, there is only one perpendicular line to b_i. Hence, all h_k's are scalar multiples of each other. The second part of Assumption 14 implies that they are indeed positive multiples of each other. In other words, all the semicircles $\{y \in \mathbb{S}^1 | h_k^T y \leq 0\}$ for $k \in J(o_i) \cap J(o_{i+1})$ are the same, and equal exactly one of the arcs $\mathcal{K}_1$ or $\mathcal{K}_2$.

The next two lemmas will provide a connection between

f_i's, $\tilde{f}_i$'s and g_i's. This will lead to an easily checkable characterization of Problem 2 in terms of null-homotopic loops.

Lemma 16: The paths f_i and $\tilde{f}_i$ are homotopic.

Proof: Let $\overline{\tilde{f}_i}$ be the reverse path of $\tilde{f}_i$. We will show that $f_i\overline{\tilde{f}_i}$ is null-homotopic. First, f_i and $\tilde{f}_i$ have the same start and end points, so $f_i\overline{\tilde{f}_i}$ is a loop in $\mathbb{S}^1$. For any $x \in \overline{o_io_{i+1}}$, $I(x) \subset I(o_i) \cup I(o_{i+1})$ so $J(x) \supset J(o_i) \cap J(o_{i+1})$. By assumption, $f_i(x) \in \mathcal{C}(x)$, $x \in \overline{o_io_{i+1}}$, so $h_j \cdot f_i(x) \leq 0$, $x \in \overline{o_io_{i+1}}$, $j \in J(x) \supset J(o_i) \cap J(o_{i+1})$. Now consider $\tilde{f}_i(x)$, $x \in \overline{o_io_{i+1}}$. Since it is the shorter arc from $F(o_i)$ to $F(o_{i+1})$, $\tilde{f}_i(x)$ is a positive multiple of a convex combination of $F(o_i)$ and $F(o_{i+1})$. Since $F(o_i) \in \mathcal{C}(o_i)$ and $F(o_{i+1}) \in \mathcal{C}(o_{i+1})$, we get $h_j \cdot \tilde{f}_i(x) \leq 0$, $x \in \overline{o_io_{i+1}}$, $j \in J(x) \supset J(o_i) \cap J(o_{i+1})$. By Lemma 12, there exists $k \in J(o_i) \cap J(o_{i+1})$. We conclude $f_i\overline{\tilde{f}_i} \subset \{y \in \mathbb{S}^1 \mid h_k \cdot y \leq 0\}$. This implies $f_i\overline{\tilde{f}_i}$ is not surjective. By Lemma 7, it is null-homotopic. By Lemma 10, $f_i \simeq \tilde{f}_i$, as desired. ∎

Lemma 17: The paths $\tilde{f}_i m_{i+1}\overline{g}_i$ and m_i are homotopic.

Proof: Both $\tilde{f}_i m_{i+1}\overline{g}_i$ and m_i are paths from $F(o_i)$ to b_{i+1}. Thus, we will prove that the loop $\tilde{f}_i m_{i+1}\overline{g}_i\overline{m}_i$ is null-homotopic. We showed in the proof above that there exists $k \in J(o_i) \cap J(o_{i+1})$ such that $h_k \cdot y \leq 0$ for all y in $\tilde{f}_i$. Also, $h_k \cdot y \leq 0$ for all y in g_i since $b_i \in \mathcal{C}(o_i)$, $b_{i+1} \in \mathcal{C}(o_{i+1})$, and any y in g_i is a positive scalar multiple of a convex combination of b_i and b_{i+1}. Next consider m_i. Since $b_i, F(o_i) \in \mathcal{C}(o_i)$, we have $h_k \cdot b_i \leq 0$, $h_k \cdot F(o_i) \leq 0$. Then, since every y in m_i is a positive multiple of a convex combination of b_i and $F(o_i)$, we get $h_k \cdot y \leq 0$ for all y in m_i. By an analogous argument we find $h_k \cdot y \leq 0$ for all y in m_{i+1}. We conclude that $h_k \cdot y \leq 0$ for all y in $\tilde{f}_i m_{i+1}\overline{g}_i\overline{m}_i$. This implies $\tilde{f}_i m_{i+1}\overline{g}_i\overline{m}_i$ is not surjective so by Lemma 7, it is null-homotoic. By Lemma 10, $\tilde{f}_i m_{i+1}\overline{g}_i \simeq m_i$, as desired. ∎

We now present our main technical tool for characterizing the topological obstruction. We note that indices are taken modulo r, i.e., $o_{r+1} \equiv o_1$. Recall that, for any $b_i, b_{i+1} \in \mathbb{S}^1$, g_i is defined as the shorter arc in $\mathbb{S}^1$ between b_i and b_{i+1}.

Proposition 18 solves Problem 2 in the case of a two-dimensional polytope $\mathcal{A}$ by exploring the homotopy classes of maps defined on its boundary ∂A. This will serve as the foundation of our final result given in Theorem 19.

Proposition 18: Let Assumption 14 hold, $m = 2$, and $\kappa \geq 2$. Let $\mathcal{A} = \text{co}\{o_1, \cdots, o_r\}$ be a two-dimensional polytope in $\partial_2\mathcal{O}_\mathcal{S}$. Then there exists $F : \mathcal{A} \to \mathbb{S}^1$ such that $F(x) \in \mathcal{C}(x)$, $x \in \mathcal{A}$, if and only if

(i) There exists $\{b_1, \cdots, b_r\}$ with $b_i \in \mathbb{S}^1 \cap \mathcal{C}(o_i)$, $i = 1, \cdots, r$.

(ii) For any selection $\{b_1, \cdots, b_r \mid b_i \in \mathbb{S}^1 \cap \mathcal{C}(o_i)\}$, the map $g : \partial A \to \mathbb{S}^1$ defined by $g(o_i) = b_i$, $i = 1, \cdots, r$ and $g|_{\overline{o_io_{i+1}}} := g_i$ is null-homotopic.

Proof: ($\Longleftarrow$) Suppose (i)-(ii) are satisfied. Let g be as in (ii). We claim g satisfies $g(x) \in \mathcal{C}(x)$, $x \in \partial\mathcal{A}$. First, $g(o_i) = b_i \in \mathcal{C}(o_i)$, $g(o_{i+1}) = b_{i+1} \in \mathcal{C}(o_{i+1})$ by (ii). Second, let $x \in \partial\mathcal{A}\backslash\{o_1, \cdots, o_r\}$. Then x is in the relative interior of $\overline{o_io_{i+1}}$ for some i. We know $I(x) = I(o_i) \cup I(o_{i+1})$, so $J(x) = J(o_i) \cap J(o_{i+1})$. Since g_i is on the shorter arc between b_i and b_{i+1}, $g(x)$ is a positive multiple of a convex combination of b_i and b_{i+1}. Since $h_j \cdot b_i \leq 0$, $j \in J(o_i)$, $h_j \cdot b_{i+1} \leq 0$, $j \in J(o_{i+1})$, then $h_j \cdot g(x) \leq 0$, $j \in J(o_i) \cap J(o_{i+1})$, so $g(x) \in \mathcal{C}(x)$.

By assumption, g is null-homotopic. Now, if $\text{Im}(g) = \mathbb{S}^1$, g can be extended to $F : \mathcal{A} \to \text{Im}(g)$ by Theorem 11. If g is not surjective, its image is a circular arc. As an arc is contractible, by Lemma 8, $g : \mathcal{A} \to \text{Im}(g)$ is null-homotopic. Hence, by Theorem 11, it can again be extended to $F : \mathcal{A} \to \text{Im}(g)$.

We claim $F(x) \in \mathcal{C}(x)$, $x \in \mathcal{A}$. For $x \in \partial\mathcal{A}$, we have $F(x) = g(x) \in \mathcal{C}(x)$. For $x \in \mathcal{A}^\circ$, $F(x) = g(z)$ for some $z \in \partial\mathcal{A}$. As in Lemma 13, $\mathcal{C}(z) \subset \mathcal{C}(x)$. Thus, $F(x) = g(z) \in \mathcal{C}(z) \subset \mathcal{C}(x)$.

($\Longrightarrow$) For the converse direction, suppose there exists $F : \mathcal{A} \to \mathbb{S}^1$ such that $F(x) \in \mathcal{C}(x)$, $x \in \mathcal{A}$. Then (i) is immediately satisfied by taking $b_i := F(o_i) \in \mathbb{S}^1 \cap \mathcal{C}(o_i)$, $i = 1, \cdots, r$. To prove (ii), let $g : \partial\mathcal{A} \to \mathbb{S}^1$ be any map such that $g(o_i) = b_i \in \mathcal{C}(o_i)$, $i = 1, \cdots, r$ and $g|_{\overline{o_io_{i+1}}} = g_i$, the shorter arc in $\mathbb{S}^1$ between b_i and b_{i+1}. We will show that g is null-homotopic. First, we claim that $F|_{\partial\mathcal{A}}$ and g are homotopic. Since $F|_{\partial\mathcal{A}}$ extends to F on $\mathcal{A}$, by Theorem 11 $F|_{\partial\mathcal{A}}$ is null-homotopic. Then if $F|_{\partial\mathcal{A}} \simeq g$, g is also null-homotopic. Therefore, we must only show $F|_{\partial\mathcal{A}} \simeq g$.

To that end, recall that $f_i := F|_{\overline{o_io_{i+1}}}$ and $\tilde{f}_i$ denotes the shorter arc in $\mathbb{S}^1$ between $F(o_i)$ and $F(o_{i+1})$. Define $\tilde{f} : \partial\mathcal{A} \to \mathbb{S}^1$ by $\tilde{f}|_{\overline{o_io_{i+1}}} = \tilde{f}_i$, $i = 1, \cdots, r$, with indices again taken modulo r. Also define $g : \partial\mathcal{A} \to \mathbb{S}^1$ to be the concatenation of g_i, $i = 1, \cdots, r$. In our loop notation, $F|_{\partial\mathcal{A}} = f_1 \cdots f_r$, $g = g_1 \cdots g_r$, and $\tilde{f} = \tilde{f}_1 \cdots \tilde{f}_r$.

By Lemma 16, $f_i \simeq \tilde{f}_i$ so by iterating on Proposition 7.10 of [18], $F|_{\partial\mathcal{A}} = f_1 \cdots f_r \simeq \tilde{f}_1 \cdots \tilde{f}_r = \tilde{f}$. Now consider $\tilde{f}_1 \cdots \tilde{f}_r m_1 \overline{g}_r \cdots \overline{g}_1 \overline{m}_1$. By Lemma 17, $\tilde{f}_r m_1 \overline{g}_r \simeq m_r$, so $\tilde{f}_1 \cdots \tilde{f}_r m_1 \overline{g}_r \cdots \overline{g}_1 \overline{m}_1 \simeq \tilde{f}_1 \cdots \tilde{f}_{r-1} m_r \overline{g}_{r-1} \cdots \overline{g}_1 \overline{m}_1$. Iterating on this argument, we get $\tilde{f} m_1 \overline{g} \overline{m}_1 \simeq \tilde{f}_1 m_2 \overline{g}_1 \overline{m}_1$. Again by Lemma 17, $\tilde{f}_1 m_2 \overline{g}_1 \overline{m}_1 \simeq m_1 \overline{m}_1$, and by Lemma 10, that path is null-homotopic. Thus, $\tilde{f} m_1 \overline{g m}_1$ is null-homotopic. Now, $m_1 g \overline{m}_1$ is a loop from $F(o_1)$ through $b_1, \cdots, b_r$ back to $F(o_1)$. Equivalently, it can be expressed as $g\overline{m}_1 m_1$, a loop starting and ending at b_1. By Lemma 10, $g\overline{m}_1 m_1 \overline{g} = g\overline{m}_1 \overline{g\overline{m}_1}$ is null-homotopic. Again applying Lemma 10, this implies $g\overline{m}_1 m_1 \simeq g$. We conclude $\tilde{f} \simeq m_1 g \overline{m}_1 \simeq g$. We already showed $F|_{\partial\mathcal{A}} \simeq \tilde{f}$. We conclude $F|_{\partial\mathcal{A}} \simeq g$, as desired. ∎

Using Lemma 13 and Proposition 18, we are now ready to prove our main result.

Theorem 19: Let Assumption 14 hold and let $\dim(\mathcal{B}) = 2$. There exists a continuous function $F : \mathcal{O}_\mathcal{S} \to \mathcal{B}\backslash\{0\}$ such that $F(x) \in \mathcal{C}(x)$, $x \in \mathcal{O}_\mathcal{S}$, if and only if:

(i) For all $o_i \in V_{\mathcal{O}_\mathcal{S}}$, there exists $b_i \in \mathcal{B} \cap \mathcal{C}(o_i)$, $b_i \neq 0$.

(ii) For every two-dimensional polytope $\mathcal{A} = \text{co}\{o_1, \cdots, o_r\}$ in $\partial_2\mathcal{O}_\mathcal{S}$ and for any selection $\{b_1, \cdots, b_r \mid b_i \in \mathcal{B} \cap \mathcal{C}(o_i), b_i \neq 0\}$, the map $g : \partial\mathcal{A} \to \mathcal{B}\backslash\{0\} \simeq \mathbb{S}^1$ defined by $g(o_i) = b_i$, $i = 1, \cdots, r$ and $g|_{\overline{o_io_{i+1}}} := g_i$ is null-homotopic.

Proof: First we consider the case that $\dim(\mathcal{O}_{\mathcal{S}}) = \kappa \geq 2$. Suppose (i)-(ii) hold. By Proposition 18, for every 2-dimensional $\mathcal{A}$ there exists a function $F_{\mathcal{A}} : \mathcal{A} \to \mathbb{S}^1$ satisfying (3). We also know that all functions $F_{\mathcal{A}}$ agree on the 1-dimensional edges of $\mathcal{O}_{\mathcal{S}}$, as $F_{\mathcal{A}}|_{\partial A} \equiv g$, and the definition of g on some edge $\overline{o_i o_j}$ of $\mathcal{O}_{\mathcal{S}}$ only depends on b_i and b_j, i.e., does not depend on $\mathcal{A}$. Hence, all functions $F_{\mathcal{A}}$ can be "glued" together into a continuous function $F : \partial_2 \mathcal{O}_{\mathcal{S}} \to \mathbb{S}^1 \simeq \mathcal{B}\backslash\{0\}$ satisfying (3). By Lemma 13, F can be extended to a function $F : \mathcal{O}_{\mathcal{S}} \to \mathcal{B}\backslash\{0\}$ satisfying $F(x) \in \mathcal{C}(x)$, $x \in \mathcal{O}_{\mathcal{S}}$.

In the converse direction, suppose $\kappa \geq 2$ and there exists a function $F : \mathcal{O}_{\mathcal{S}} \to \mathcal{B}\backslash\{0\}$ such that $F(x) \in \mathcal{C}(x)$, $x \in \mathcal{O}_{\mathcal{S}}$. Then (i) is automatically satisfied by taking $b_i = F(o_i)$. For (ii), we note that the function $F|_{\mathcal{A}}$ satisfies the Proposition 18. Hence, (i)-(ii) in Proposition 18 hold, and by (ii), g is null-homotopic.

The only case remaining is $\kappa = 1$. In that case, (ii) is vacuous, i.e., there are no two-dimensional polytopes $\mathcal{A}$. Now assume that (i) holds. Let $\mathcal{O}_{\mathcal{S}} = \overline{o_1 o_2}$. We define $F : \mathcal{O}_{\mathcal{S}} \to \mathbb{S}^1 \simeq \mathcal{B}\backslash\{0\}$ as the shorter arc connecting b_1 and b_2. If the length of arc between b_1 and b_2 is exactly π, by Assumption 14 and our discussion in Remark 15, we know that at least one of those two arcs lies in $\{y \in \mathbb{S}^1 | \tilde{h}_j y \leq 0\}$, for all $j \in J(o_1) \cap J(o_2)$. We choose that arc as the "shorter". (The set $J(o_1) \cap J(o_2)$ may be empty, in which case both of those arcs satisfy our conditions.) By the same discussion as in the proof of Proposition 18, we know that F so defined satisfies (3) on all of $\mathcal{O}_{\mathcal{S}}$, and hence solves Problem 2. On the other hand, if we assume that there exists a function $F : \mathcal{O}_{\mathcal{S}} \to \mathcal{B}\backslash\{0\}$ satisfying $F(x) \in \mathcal{C}(x)$, $x \in \mathcal{O}_{\mathcal{S}}$, (i) is automatically satisfied by taking $b_i = F(o_i)$. ■

V. Conclusion

This paper expands on the notion of a topological obstruction to solving the RCP by continuous state feedback introduced in [21]. Theorem 19 solves this problem in the case of systems with two inputs. The results require an inspection of cones $\mathcal{C}(x)$ at the vertices of $\mathcal{O}_{\mathcal{S}}$, as well as a check of whether the function g is null-homotopic. We note that Theorem 19 also requires technical Assumption 14 to work. This assumption can be removed, but the resulting necessary and sufficient conditions are less elegant and we omit them in the interest of space and readability.

The most immediate obstacle to generalizing the results to the case of $m > 2$ is in Lemma 13. As mentioned, it has been shown in [17] that it is not generally true that any continuous function $f : \mathbb{S}^k \to \mathbb{S}^l$ with $k > l \geq 2$ is null-homotopic. Hence, an analogue to Lemma 13 in higher dimensions does not exist.

However, there is a possible alternative approach. It is not difficult to check that Lemma 12 can essentially be generalized to edges of $\mathcal{O}_{\mathcal{S}}$ of higher dimensions. Hence, for any edge $\mathcal{A}$ in $\partial_{\kappa-1}\mathcal{O}_{\mathcal{S}}$, every continuous function $f_{\mathcal{A}} : \mathcal{A} \to \mathbb{S}^{m-1}$ that satisfies the cone invariance conditions is not surjective, and hence null-homotopic. Thus, the main problem in extending the results of this paper is finding the conditions to "glue" the null-homotopic functions $f_{\mathcal{A}}$ into a null-homotopic function $\partial_{\kappa-1} f : \partial_{\kappa-1}\mathcal{O}_{\mathcal{S}} \to \mathbb{S}^{m-1}$.

References

[1] G. Ashford and M. E. Broucke. Time-varying affine feedback for each control on simplices. *Automatica*, vol. 49, issue 5, pp. 1365-1369, May 2013.

[2] G. Ashford and M. E. Broucke. Design of reach controllers on simplices. *IEEE Conference on Decision and Control.* December 2013.

[3] W. F. Basener. *Topology and Its Applications*. Wiley, 2006.

[4] C. Belta, V. Isler, and G. J. Pappas. Discrete abstractions for robot motion planning and control in polygonal environments. *IEEE Transactions on Robotics*. Vol. 21, no. 5, pp. 864–874, 2005.

[5] C. Belta, L. C. G. J. M. Habets. Controlling a class of nonlinear systems on rectangles. *IEEE Transactions on Automatic Control*. Vol. 51, no. 11, pp. 1749–1759, 2006.

[6] C. Belta, L. C. G. J. M. Habets, and V. Kumar. Control of multi-affine systems on rectangles with applications to biomolecular networks. *IEEE Conference on Decision and Control*. pp. 534–539, 2002.

[7] G. E. Bredon. *Topology and Geometry*. Springer, 1997.

[8] R. W. Brockett. Asymptotic stability and feedback stabilization, in: *Differential Geometric Control Theory*. R. W. Brockett, R. S. Millman and H. J. Sussmann, eds., Birkhauser, 1983.

[9] M. E. Broucke. Reach control on simplices by continuous state feedback. *SIAM Journal on Control and Optimization*. Vol. 48, no. 5, pp. 3482–3500, February 2010.

[10] M. E. Broucke and M. Ganness. Reach control on simplices by piecewise affine feedback. *SIAM Journal on Control and Optimization.* Vol. 52, no. 5, pp. 3261-3286, October 2014.

[11] M. Ganness. *Reach Control on Simplices by Piecewise Affine Feedback*. M.A.Sc. thesis, Department of Electrical and Computer Engineering, University of Toronto, 2010.

[12] L. C. G. J. M. Habets and J. H. van Schuppen. A control problem for affine dynamical systems on a full-dimensional polytope. *Automatica*. no. 40, pp. 21–35, 2004.

[13] L. C. G. J. M. Habets, P. J. Collins, and J. H. van Schuppen. Reachability and control synthesis for piecewise-affine hybrid systems on simplices. *IEEE Transactions on Automatic Control*. no. 51, pp. 938–948, 2006.

[14] A. Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.

[15] S. Haugwitz and P. Hagander. Analysis and design of startup control of a chemical plate reactor with uncertainties: a hybrid approach. *IEEE Conference on Control Applications*. pp. 1426–1431, 2007.

[16] M. K. Helwa and M. E. Broucke. Reach control of single-input systems on simplices using multi-affine feedback. *Mathematical Theory of Networks and Systems*. pp. 748–755, 2014.

[17] H. Hopf. Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche. *Mathematische Annalen*. Vol. 104, no. 1, pp. 637–665, 1931.

[18] J. Lee. *Introduction to Topological Manifolds*. Springer, 2010.

[19] M. Martino and M. E. Broucke. A reach control approach to bumpless transfer of robotic manipulators. *IEEE Conference on Decision and Control*. pp. 25–30, 2014.

[20] G. L. Naber. *Topological Methods in Euclidean Spaces*. Dover, 2000.

[21] M. Ornik and M. E. Broucke. On a Topological Obstruction in the Reach Control Problem. *Proceedings of AMMCS-CAIMS Congress*. Submitted, July 2015.

[22] B. Roszak and M. E. Broucke. Necessary and sufficient conditions for reachability on a simplex. *Automatica*. Vol. 42, no. 11, pp. 1913–1918, 2006.

[23] E. Semsar-Kazerooni and M. E. Broucke. Reach controllability of single input affine systems. *IEEE Transactions on Automatic Control*. Vol. 59, no. 3, pp. 738–744, 2014.

[24] M. Vukosavljev and M. E. Broucke. Control of a gantry crane: a reach control approach. *IEEE Conference on Decision and Control*. pp. 3609–3614, 2014.

[25] T. Wongpiromsarn, U. Topcu, and R. M. Murray. Receding horizon temporal logic planning. *IEEE Transactions on Automatic Control*. vol. 57, no. 11, pp. 2817–2830, 2012.