Some Results on an Affine Obstruction to Reach Control

Melkior Ornik*, Mireille E. Broucke*

July 10, 2015

Abstract

This ArXiv paper is a supplement to [7] and contains proofs of preliminary claims omitted in [7] for lack of space. The paper deals with exploring a necessary condition for solvability of the Reach Control Problem (RCP) using affine feedback. The goal of the RCP is to drive the states of an affine control system to a given facet of a simplex without first exiting the simplex through other facets. In analogy to the problem of a topological obstruction to the RCP for continuous state feedback studied in [7], this paper formulates the problem of an affine obstruction and solves it in the case of two- and three-dimensional systems. An appealing geometric cone condition is identified as the new necessary condition.

1 Introduction

The Reach Control Problem (RCP), first introduced in [4] and given a modern formulation in [5, 9], is a fundamental problem in piecewise affine and hybrid system theory. A reach control approach has been shown to be useful in a number of applications, including aircraft and underwater vehicles [1], genetic networks [2], and aggressive maneuvers of mechanical systems [12]. Nevertheless, for a given system, it is still not known in general whether the RCP is solvable by either affine or continuous state feedback. This paper formulates the problem of an obstruction to solving the RCP by affine state feedback.

Consider an n-dimensional simplex $\mathcal{S} \subseteq \mathbb{R}^n$ with vertices $v_0, v_1, \ldots, v_n$. The facets of $\mathcal{S}$ are denoted by $\mathcal{F}_0, \ldots, \mathcal{F}_n$, where each facet is indexed by the vertex it does not contain. The facet $\mathcal{F}_0$ is called the *exit facet*. We consider an affine control system defined on $\mathcal{S}$, given by

$$\dot{x} = Ax + Bu + a\,. \tag{1.1}$$

The RCP asks the following question: Is it possible to find a state feedback $u : \mathcal{S} \to \mathbb{R}^m$ such that, for any initial state $x_0 \in \mathcal{S}$, the closed-loop trajectory leaves $\mathcal{S}$ in finite time, and it does so by leaving through facet $\mathcal{F}_0$? Let $\phi_u(t, x_0)$ be the trajectory of system (1.1) under state feedback u and with initial state x_0.

*Department of Electrical and Computer Engineering, University of Toronto, Toronto ON Canada M5S 3G4, {melkior.ornik@scg.utoronto.ca, broucke@control.utoronto.ca}

Problem 1. *Is it possible to find $u : \mathcal{S} \to \mathbb{R}^m$ such that for each $x_0 \in \mathcal{S}$, there exists $T > 0$ such that*

(i) $\phi_u(t, x_0) \in \mathcal{S}$ *for all* $t \in [0, T]$,

(ii) $\phi_u(T, x_0) \in \mathcal{F}_0$,

(iii) $\phi_u(t, x_0) \notin \mathcal{S}$ *for all* $t \in (T, T + \varepsilon)$ *for some* $\epsilon > 0$.

In this paper we focus on a necessary condition for solvability of the RCP using affine feedback. In particular, for an affine state feedback to solve the RCP, it must not admit any closed-loop equilibria in $\mathcal{S}$. Let $\mathcal{B} = \mathrm{Im}(B)$. It is easily shown that the equilibria of (1.1) can only lie in the affine subspace

$$\mathcal{O} = \{x \in \mathbb{R}^n \mid Ax + a \in \mathcal{B}\}.$$

As we are only interested in potential equilibria contained in $\mathcal{S}$, we study the set

$$\mathcal{O}_{\mathcal{S}} = \mathcal{S} \cap \mathcal{O}.$$

We are interested in seeing whether we can design an affine feedback on $\mathcal{O}_{\mathcal{S}}$ satisfying the conditions of Problem 1. This is clearly a necessary condition for solvability of the RCP.

For each $x \in \mathcal{S}$, we define the inside pointing cone $\mathcal{C}(x)$ with respect to $\mathcal{S}$ by

$$\mathcal{C}(x) = \{y \in \mathbb{R}^n | h_j \cdot y \leq 0 \text{ for all } j \in \{1, \ldots, n\} \backslash I(x)\}, \tag{1.2}$$

where h_j is an outward pointing normal to $\mathcal{F}_j$, and $I(x) = \{i_1, i_2, \ldots, i_k\}$ is the minimal set of indices of vertices of $\mathcal{S}$ such that $x \in \mathrm{co}\{v_{i_1}, \ldots, v_{i_k}\}$. In other words, x is in the interior of $x \in \mathrm{co}\{v_{i_1}, \ldots, v_{i_k}\}$.

$\mathcal{C}(x)$ contains all vector directions that, when appended to x, point inside $\mathcal{S}$ or through $\mathcal{F}_0$. Hence, by the conditions of Problem 1, $f(x) = Ax + Bu(x) + a \in \mathcal{C}(x)$ for all $x \in \mathcal{S}$ is a necessary condition for solvability of the RCP. If u is an affine state feedback, f is affine as well. We also note that for $x \in \mathcal{O}_{\mathcal{S}}$, $Ax + a \in \mathcal{B}$, and hence, $f(x) \in \mathcal{B}$ for $x \in \mathcal{O}_{\mathcal{S}}$.

Thus, given the above observations, we want to study the following problem:

Problem 2. *Let $\mathcal{O}_{\mathcal{S}}$, $\mathcal{S}$ and $\mathcal{B}$ be as above. Does there exist an affine map $f : \mathcal{O}_{\mathcal{S}} \to \mathcal{B}$ that satisfies*

(i) $f(x) \in \mathcal{C}(x)$ *for all* $x \in \mathcal{O}_{\mathcal{S}}$,

(ii) $f(x) \neq 0$ *for all* $x \in \mathcal{O}_{\mathcal{S}}$*?*

The second condition implies that the system given by $\dot{x} = f(x)$ contains no equilibria in $\mathcal{S}$. This problem has a continuous analogue studied in [7, 8]. The continuous analogue is referred to as the *topological obstruction problem in the RCP*, as stated below.

Problem 3. *Let $\mathcal{O}_{\mathcal{S}}$, $\mathcal{S}$ and $\mathcal{B}$ be as above. Does there exist a continuous map $f : \mathcal{O}_{\mathcal{S}} \to \mathcal{B} \backslash \{0\}$ such that, for every $x \in \mathcal{O}_{\mathcal{S}}$, $f(x) \in \mathcal{C}(x)$?*

Section 2 of this paper contains preliminary results for several special cases. These apply equally to Problem 2 and to Problem 3. Hence, this paper serves as a supplement to [7], providing simple proofs omitted in [7] for lack of space. In Section 3, we provide a solution to Problem 2 in the cases of $n = 2$ and $n = 3$ using a linear algebra approach.

2 Preliminaries

In this section we introduce a sufficient condition for solvability of Problems 2 and 3. Furthermore, we investigate the cases of $\dim \mathcal{O}_{\mathcal{S}} = 0$ and $\dim \mathcal{O}_{\mathcal{S}} = n$. All of the following results apply both to Problem 2 and 3.

Lemma 4 (Vertex Deletion)**.** *Let $I(p) = \{0, i_1, i_2, \ldots, i_k\}$, with $k \geq 0$. Furthermore, let $I(q) = \{i_1, i_2, \ldots, i_k, i_{k+1}, \ldots, v_{i_l}\}$, where $l \geq k$. We take all i_j's to be different, and all different from 0. Then $\mathcal{C}(p) \subseteq \mathcal{C}(q)$.*

Proof. By the definition of $\mathcal{C}$ in (1.2),

$$\mathcal{C}(p) = \{y \in \mathbb{R}^n | h_j \cdot y \leq 0 \text{ for all } j \in \{1, \ldots, n\} \backslash \{0, i_1, i_2, \ldots, i_k\}\}.$$

On the other hand,

$$\mathcal{C}(q) = \{y \in \mathbb{R}^n | h_j \cdot y \leq 0 \text{ for all } j \in \{1, \ldots, n\} \backslash \{i_1, i_2, \ldots, i_k, i_{k+1}, \ldots, i_l\}\}.$$

Thus, as it is clear that the set of constraints in $\mathcal{C}(q)$ is a subset of the set of constraints in $\mathcal{C}(p)$, $\mathcal{C}(p) \subseteq \mathcal{C}(q)$. □

Remark 5. *From the proof, it is clear that it does not matter if $I(p)$ includes 0 or not. Analogously, it does not matter if q is in the convex hull of vertices that include v_0 or not.*

The above lemma can now be used to show that cones of points on the interior of a polytope in $\mathcal{O}_{\mathcal{S}}$ are less restrictive than cones of points at its boundary. This is given in Lemma 6, and such a claim will be useful both for Problem 2 discussed in Section 3, as well as in Problem 3 discussed in [7].

Lemma 6. *Let $\mathcal{H} \subseteq \mathcal{O}_{\mathcal{S}}$ be a polytope, and let x be any point in its interior: $x \in \operatorname{Int}(\mathcal{H})$. Also, let y be any point on its boundary: $y \in \partial\mathcal{H}$. Then, $\mathcal{C}(y) \subseteq \mathcal{C}(x)$.*

Proof. Consider the line going through points x and y. As $x \notin \partial\mathcal{H}$, by extending that line past x, we can determine a point $z \in \partial\mathcal{H}$ such that $x = \alpha y + \beta z$, where $\alpha, \beta > 0$, $\alpha + \beta = 1$. Let us assume that $y = \sum_{i=0}^{n} \alpha_i v_i$, $z = \sum_{i=0}^{n} \beta_i v_i$. Since both y and z are in $\mathcal{O}_{\mathcal{S}} \subseteq \mathcal{S}$, all α_i's and β_i's are nonnegative. Then, $x = \sum_{i=0}^{n} (\alpha\alpha_i + \beta\beta_i) v_i$. Now, for any i, if $i \in I(y)$, then $\alpha_i \neq 0$. We notice that, in that case, no matter what β, β_i and α are, $\alpha\alpha_i + \beta\beta_i > 0$. Thus, $i \in I(x)$. In other words, $I(y) \subseteq I(x)$ and thus, by Lemma 4, $\mathcal{C}(y) \subseteq \mathcal{C}(x)$. □

From now on, we will use the following notation:

$$\operatorname{cone}(\mathcal{O}_{\mathcal{S}}) = \bigcap_{x \in \mathcal{O}_{\mathcal{S}}} \mathcal{C}(x).$$

We note that by Lemma 6

$$\operatorname{cone}(\mathcal{O}_{\mathcal{S}}) = \bigcap_{i=1}^{r} \mathcal{C}(o_i),$$

where $o_1, \dots, o_r$ are vertices of $\mathcal{O}_\mathcal{S}$.

The following result provides a sufficient condition for solving Problems 2 and 3. Our discussion in Section 3 will show that this condition is not necessary in general. However, it holds a central position in treatment of a number of subcases when solving Problems 2 and 3 for $n = 2, 3$.

Lemma 7. *If*

$$\text{cone}(\mathcal{O}_\mathcal{S}) \cap \mathcal{B} \neq \{0\},$$

then the answer to Problems 2 and 3 is affirmative.

Proof. Let $b \in \text{cone}(\mathcal{O}_\mathcal{S}) \cap \mathcal{B} \backslash \{0\}$. We note that, by definition of $\text{cone}(\mathcal{O}_\mathcal{S})$, b satisfies the inward-pointing condition at every point in $\mathcal{O}_\mathcal{S}$. Thus, the function $f : \mathcal{O}_\mathcal{S} \to \mathcal{B}\backslash\{0\}$ defined by $f(x) = b$ for all $x \in \mathcal{O}_\mathcal{S}$ satisfies all the criteria of Problem 2 and of Problem 3. □

As a dual of sorts to Lemma 7, Lemma 8 gives an easy necessary condition for Problems 2 and 3.

Lemma 8. *Assume that the function f from Problem 2 (Problem 3) exists. Then, for every $x \in \mathcal{O}_\mathcal{S}$, there exists $0 \neq b \in \mathcal{C}(x) \cap \mathcal{B}$.*

Proof. For any such x, take $b = f(x)$. By the conditions of Problems 2 and 3, $f(x) \in \mathcal{B}\backslash\{0\}$ and $f(x) \in \mathcal{C}(x)$. □

Finally, let us note that $\mathcal{O}_\mathcal{S}$ is a manifold (in fact, a polytope) of dimension $0 \leq \dim \mathcal{O}_\mathcal{S} \leq n$. Cases $\dim \mathcal{O}_\mathcal{S} = 0$ and $\dim \mathcal{O}_\mathcal{S} = n$, as well as the case of $v_0 \in \mathcal{O}_\mathcal{S}$, prove to be particularly easy to analyze. We do that as follows:

Lemma 9. *If $\dim \mathcal{O}_\mathcal{S} = 0$, the answer to Problems 2 and 3 is affirmative if and only if*

$$\text{cone}(\mathcal{O}_\mathcal{S}) \cap \mathcal{B} \neq \mathbf{0}.$$

Proof. We note that in this case, $\mathcal{O}_\mathcal{S}$ consists of a single point $x \in \mathcal{S}$. Thus, $\text{cone}(\mathcal{O}_\mathcal{S}) = \mathcal{C}(x)$, sufficiency is proved by Lemma 7, and necessity is proved by Lemma 8. □

Lemma 10. *If $v_0 \in \mathcal{O}_\mathcal{S}$, then $\text{cone}(\mathcal{O}_\mathcal{S}) = \mathcal{C}(v_0)$ and the answer to Problems 2 and 3 is affirmative if and only if*

$$\text{cone}(\mathcal{O}_\mathcal{S}) \cap \mathcal{B} = \mathcal{C}(v_0) \cap \mathcal{B} \neq \mathbf{0}.$$

Proof. Sufficiency is proved by Lemma 7. Now, by the Vertex Deletion Lemma, $\mathcal{C}(v_0) \subseteq \mathcal{C}(x)$ for all $x \in \mathcal{O}_\mathcal{S}$. Thus,

$$\mathcal{C}(v_0) \supseteq \text{cone}(\mathcal{O}_\mathcal{S}) = \bigcap_{x \in \mathcal{O}_\mathcal{S}} \mathcal{C}(x) \supseteq \text{cone}(v_0).$$

So, $\text{cone}(\mathcal{O}_\mathcal{S}) = \mathcal{C}(v_0)$, and necessity thus follows from Lemma 8. □

Corollary 11. *If $\dim \mathcal{O}_\mathcal{S} = n$, then $\text{cone}(\mathcal{O}_\mathcal{S}) = \mathcal{C}(v_0)$ and the answer to Problems 2 and 3 is affirmative if and only if $\text{cone}(\mathcal{O}_\mathcal{S}) \cap \mathcal{B} = \mathcal{C}(v_0) \cap \mathcal{B} \neq \mathbf{0}$.*

Proof. We note that $\dim \mathcal{O}_\mathcal{S} = n$ implies $\mathcal{O}_\mathcal{S} = \mathcal{S} \ni v_0$. The claim follows from Lemma 10. □

In the remainder of the text, as well as in [7], we assume that $1 \leq \dim \mathcal{O}_\mathcal{S} \leq n - 1$.

3 Affine Case

This section contains the main contribution of this paper: we will solve Problem 2 in the case of $n = 2, 3$. We will do that on a case by case basis, employing methods from linear algebra. We note that the results from the previous section solved the cases of $\dim \mathcal{O}_\mathcal{S} \in \{0, n\}$. This reduces the problem to $\dim \mathcal{O}_\mathcal{S} = 1$ and $\dim \mathcal{O}_\mathcal{S} = 2$ (when $n = 3$). In both of these, the sufficient condition

$$\operatorname{cone}(\mathcal{O}_\mathcal{S}) \cap \mathcal{B} \neq \mathbf{0}$$

from Lemma 7 will again make an appearance, as it will be shown that, depending on the case, Problem 2 is either always solvable, or the condition from Lemma 7 is a necessary condition.

3.1 $n = 2$

We note that the case where $\dim \mathcal{O}_\mathcal{S} \in \{0, 2\}$ has been solved in Lemma 9 and Corollary 11. Thus, the only remaining case is when $\dim \mathcal{O}_\mathcal{S} = \dim \mathcal{B} = 1$. However, this was covered in Theorem 1 of [10]: the same Intermediate Value Theorem argument holds for both continuous and affine functions. Thus, f from Problem 2 exists if and only if $\operatorname{cone}(\mathcal{O}_\mathcal{S}) \cap \mathcal{B} \neq \mathbf{0}$.

3.2 $n = 3$

Again, the cases in which $\dim \mathcal{O}_\mathcal{S} \in \{0, 3\}$ have been solved in Lemma 9 and Corollary 11. Thus, the remaining cases are $\dim \mathcal{O}_\mathcal{S}, \dim \mathcal{B} \in \{1, 2\}$.

$\dim \mathcal{O}_\mathcal{S} = \dim \mathcal{B} = 1$ is, as above, covered in [10]. Problem 2 is again solvable if and only if $\operatorname{cone}(\mathcal{O}_\mathcal{S}) \cap \mathcal{B} \neq \mathbf{0}$.

3.2.1 $\dim \mathcal{O}_\mathcal{S} = 1$, $\dim \mathcal{B} = 2$

The following two lemmas were stated and proved in [7]. For the benefit of the reader, we repeat the proofs. Lemma 13 solves Problem 2 in the case of $\dim \mathcal{O}_\mathcal{S} = 1$ and $\dim \mathcal{B} = 2$.

Lemma 12. *Suppose $\mathcal{O}_\mathcal{S} = \operatorname{co}\{o_1, \ldots, o_{\kappa+1}\}$ where the o_i's are the vertices of $\mathcal{O}_\mathcal{S}$. If there exists a linearly independent set $\{b_i \in \mathcal{B} \cap \mathcal{C}(o_i) \mid i = 1, \ldots, \kappa + 1\}$, then the answer to Problems 2 and 3 is affirmative.*

Proof. Let $f : \mathcal{O}_\mathcal{S} \to \mathcal{B}$ be defined by $f(\sum_{i=1}^{\kappa+1} \alpha_i o_i) = \alpha_i b_i$, where $\sum \alpha_i = 1$ and $\alpha_i \geq 0$. Necessarily $f(x) \neq 0$ for $x \in \mathcal{O}_\mathcal{S}$ for otherwise the b_i's would be linearly dependent. Also, by a standard convexity argument $f(x) \in \mathcal{C}(x)$, $x \in \mathcal{O}_\mathcal{S}$. □

Lemma 13. *Let $n = 3$, $\dim \mathcal{B} = 2$, and let o_1 and o_2 be vertices of $\mathcal{O}_\mathcal{S}$. Then there exist linearly independent vectors $\{b_1, b_2 \mid b_i \in \mathcal{B} \cap \mathcal{C}(o_i)\}$. Moreover, if $\mathcal{O}_\mathcal{S} = \operatorname{co}\{o_1, o_2\}$, the answer to Problems 2 and 3 is affirmative.*

Proof. First we assume $o_1 \in \text{ri}(\mathcal{F}_i)$ for some $i \in \{0,1,2,3\}$. By the definition of $\mathcal{C}(o_1)$, it is a closed half space or $\mathbb{R}^3$, so there exist linearly independent vectors $b_{11}, b_{12} \in \mathcal{B} \cap \mathcal{C}(o_1)$. We claim $\mathcal{B} \cap \mathcal{C}(o_2) \neq \mathbf{0}$. If $o_2 \in \text{ri}(\mathcal{F}_i)$ for some $i \in \{0,1,2,3\}$ then the argument above proves the claim. Instead, assume w.l.o.g. that $o_2 \in \mathcal{F}_1 \cap \mathcal{F}_2$. Then $\mathcal{C}(o_2) = \{y \in \mathbb{R}^3 | h_1 \cdot y \leq 0, h_2 \cdot y \leq 0\}$. Let $\mathcal{B} = \text{Ker}(M^T)$ for some $M \in \mathbb{R}^{m \times n}$. Finding $0 \neq y \mathcal{B} \cap \mathcal{C}(o_2)$ is equivalent to solving

$$\begin{bmatrix} h_1^T \\ h_2^T \\ A^T \end{bmatrix} y = \begin{bmatrix} s_1 \\ s_2 \\ 0 \end{bmatrix} \tag{3.1}$$

where $s_1, s_2 \in \mathbb{R}_0^-$ are unknown and $y \neq 0$. Because $\{h_1, h_2\}$ are linearly independent, $\text{rank}(H) \geq 2$. If $\text{rank}(H) = 3$, then let

$$[y_1 \quad y_2] = H^{-1} \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}.$$

Since $(-1,0,0)$ and $(0,-1,0)$ are linearly independent, y_1 and y_2 are linearly independent as well.

Next, assume $\text{rank}(H) = 2$. In other words, $A = c_1 h_1 + c_2 h_2$ for some $c_1, c_2 \in \mathbb{R}$. Then, by taking $s_1 = s_2 = 0$, equation (3.1) reduces to

$$\begin{bmatrix} h_1^T \\ h_2^T \end{bmatrix} y = 0.$$

Now, by the rank-nullity theorem, the dimension of the kernel of the 2×3 matrix on the left is 1. Thus, there exists a nontrivial y satisfying this equation. We make note of the fact that, if we take v_0 to be the origin, such a y satisfies $y \in \mathcal{F}_1 \cap \mathcal{F}_2 = \text{co}\{v_0, v_3\}$. In fact, since we can take any y in this intersection of planes, we can take $y = v_3$.

We are now almost done. We assumed that o_1 is in the interior of one of the facets (or on the edges of $\mathcal{F}_0$, excluding the vertices). We proved that then there exist linearly independent $b_{11}, b_{12} \in \mathcal{C}(o_1)$. We also proved that there exists a nontrivial $b_2 \in \mathcal{C}(o_2)$. We claim that at least one of the pairs $\{b_{11}, b_2\}$ and $\{b_{12}, b_2\}$ will be linearly independent. Otherwise, b_{11} is a scalar multiple of b_2, and b_2 is a scalar multiple of b_{12}. This is a contradiction with b_{11} and b_{12} being linearly independent. Thus, we have indeed found a linearly independent pair $b_1 \in \mathcal{C}(o_1)$ and $b_2 \in \mathcal{C}(o_2)$.

Let us now assume that neither o_1 nor o_2 are on the facet interiors (nor on the edges of $\mathcal{F}_0$, excluding the vertices). So, without loss of generality, $o_1 \in \mathcal{F}_1 \cap \mathcal{F}_2$ and $o_2 \in \mathcal{F}_1 \cap \mathcal{F}_3$. By the computations from several paragraphs above, we have shown that either there exist two linearly independent vectors in $\mathcal{C}(o_1)$ or $v_3 \in \mathcal{C}(o_1)$. Analogously, there either exist two linearly independent vectors in $\mathcal{C}(o_2)$ or $v_2 \in \mathcal{C}(o_2)$. Now, in the case there either exist two linearly independent vectors in $\mathcal{C}(o_1)$ or in $\mathcal{C}(o_2)$, the procedure in the previous paragraph generates a linearly independent pair of vectors, one in each of the cones.

In the remaining case, $v_3 \in \mathcal{C}(o_1)$ and $v_2 \in \mathcal{C}(o_2)$. Since v_3 and v_2 are obviously linearly independent, we again found our required pair of linearly independent vectors. Finally, if $\mathcal{O}_\mathcal{S} = \text{co}\{o_1, o_2\}$, then by Lemma 12 the answer to Problem 2 and Problem 3 is affirmative. □

This answers Problem 2 whenever $\dim \mathcal{O}_\mathcal{S} = 1$.

If $\dim \mathcal{O}_\mathcal{S} = 2$ and $\dim \mathcal{B} = 1$, the matter is clear: by the same argument in [10], which invokes the Intermediate Value Theorem, the vectors assigned at the segment between any two points B need to be positive multiples of each other. Thus, all the cones $\mathcal{C}(x)$ for $x \in \mathcal{O}_\mathcal{S}$ need to be the same. Hence, by Lemma 7 and Lemma 8, the answer to Problem 2 is affirmative if and only if $\mathrm{cone}(\mathcal{O}_\mathcal{S}) \cap \mathcal{B} \neq \mathbf{0}$.

3.2.2 $\dim \mathcal{O}_\mathcal{S} = 2$ and $\dim \mathcal{B} = 2$

We assume that $v_0 \notin \mathcal{O}_\mathcal{S}$, for that case has been settled by Lemma 10. We also assume that $\mathrm{cone}(\mathcal{O}_\mathcal{S}) \cap \mathcal{B} = \mathbf{0}$. Otherwise, we are done by Lemma 7. Now, let us observe what $\mathcal{O}_\mathcal{S}$ can look like. As given by the formula for product of simplices in [6], $\mathcal{O}_\mathcal{S}$ can either be a product of a 2-simplex and a 0-simplex, i.e., a triangle, or a product of two 1-simplices, i.e., a quadrilateral. We also must allow for $\mathcal{O}_\mathcal{S}$ passing through one of the vertices of $\mathcal{S}$, resulting in a triangle (essentially, a degenerated quadrilateral).

3.2.2.1 $\mathcal{O}_\mathcal{S}$ is a triangle

First, let us assume that $\mathcal{O}_\mathcal{S}$ satisfies $o_i \in (v_0, v_i]$ for all $i = 1, 2, 3$. Then, Theorem 15 provides a solution to both Problem 3 and Problem 2. This theorem was also stated and proved in [7], but we provide both the statement and the proof for the benefit of the reader.

We first make note of a variant of Sperner's lemma from [11]. The same variant was previously used in [3].

Lemma 14. *Let $\mathcal{P} = \mathrm{co}\{w_1, \ldots, w_{n+1}\}$ be an n-dimensional simplex. Let $\{\mathcal{Q}_1, \ldots, \mathcal{Q}_{n+1}\}$ be a collection of sets covering $\mathcal{P}$ such that*

(P1) Vertex $w_i \in \mathcal{Q}_i$ and $w_i \notin \mathcal{Q}_j$ for $j \neq i$.

(P2) If w.l.o.g. $x \in \mathrm{co}\{w_1, \ldots, w_l\}$ for some $1 \leq l \leq n+1$, then $x \in \mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_l$.

Then $\bigcap_{1=1}^{n+1} \overline{\mathcal{Q}}_i \neq \emptyset$.

Theorem 15. *Let $n = 3$ and suppose $\mathcal{O}_\mathcal{S} = \mathrm{co}\{o_1, o_2, o_3\}$ with $v_0 \notin \mathcal{O}_\mathcal{S}$ and $o_i \in (v_0, v_i]$, $i = 1, 2, 3$. The answer to Problem 3 and Problem 2 is affirmative if and only if $\mathcal{B} \cap \mathrm{cone}(\mathcal{O}_\mathcal{S}) \neq \mathbf{0}$.*

Proof. Sufficiency is provided by Lemma 7. For necessity, suppose there exists $f : \mathcal{O}_\mathcal{S} \to \mathcal{B} \setminus \{0\}$ such that $f(x) \in \mathcal{C}(x)$, $x \in \mathcal{O}_\mathcal{S}$. By way of contradiction suppose $\mathcal{B} \cap \mathrm{cone}(\mathcal{O}_\mathcal{S}) = \mathbf{0}$. Since $o_i \in (v_0, v_i]$, $i = 1, 2, 3$, we have

$$\mathrm{cone}(\mathcal{O}_\mathcal{S}) = \{y \in \mathbb{R}^n \mid h_j \cdot y \leq 0\,, j = 1, 2, 3\}\,.$$

Define the sets

$$\mathcal{Q}_i := \{x \in \mathcal{O}_\mathcal{S} \mid h_i \cdot f(x) > 0\}\,, \quad i = 1, 2, 3\,. \tag{3.2}$$

Now we verify the conditions of Lemma 14.

Firstly, we claim that $\{\mathcal{Q}_i\}$ cover $\mathcal{O}_\mathcal{S}$. For suppose not. Then there exists $x \in \mathcal{O}_\mathcal{S}$ such that $h_j \cdot f(x) \leq 0$, $j = 1, 2, 3$. Hence $f(x) \in \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{S})$, so $f(x) = 0$, a contradiction to f being non-vanishing on $\mathcal{O}_\mathcal{S}$.

Secondly, we verify property (P1). We claim that $o_i \in \mathcal{Q}_i$ for $i = 1, 2, 3$. For suppose not. Then $h_i \cdot f(x) \leq 0$. Additionally, because $f(o_i) \in \mathcal{C}(o_i)$, $h_j \cdot f(x) \leq 0$, $j \in \{1, 2, 3\} \setminus \{i\}$. We conclude $f(o_i) \in \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{S})$, so $f(o_i) = 0$, a contradiction. Next we claim $o_i \notin \mathcal{Q}_j$, $j \neq i$. This is immediate since $f(o_i) \in \mathcal{C}(o_i)$ implies $h_j \cdot f(o_i) \leq 0$, $j \neq i$.

Thirdly, we verify property (P2). Suppose w.l.o.g. (by reordering the indices $\{1, 2, 3\}$) $x \in \text{co}\{o_1, \ldots, o_r\}$ for some $1 \leq r \leq 3$. We claim $x \in \mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_r$. For suppose not. Then $h_j \cdot f(x) \leq 0$, $j = 1, \ldots, r$. Also, it is easily verified that $\mathcal{C}(x) = \{y \in \mathbb{R}^n \mid h_j \cdot y \leq 0\,, j = r+1, \ldots, 3\}$. Thus, $h_j \cdot f(x) \leq 0$, $j = r+1, \ldots, 3$. Hence, $f(x) \in \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{S})$, so $f(x) = 0$, a contradiction to f being non-vanishing on $\mathcal{O}_\mathcal{S}$.

We have verified (P1)-(P2) of Lemma 14. Applying the lemma, there exists $\overline{x} \in \bigcap_{i=1}^{3} \overline{\mathcal{Q}}_i$; that is, $h_j \cdot f(\overline{x}) \geq 0$, $j = 1, 2, 3$. We conclude that $-f(\overline{x}) \in \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{S})$, so $f(\overline{x}) = 0$, a contradiction. $\square$

From now on, we can assume that $\mathcal{O}_\mathcal{S}$ is not a triangle satisfying the conditions of Theorem 15. Thus, if $\mathcal{O}_\mathcal{S}$ is a triangle, by the discussion of simplicial products in [6], it can either pass through one of the vertices of $\mathcal{S}$, or have all its vertices on the edges of $\mathcal{S}$ which connect a single vertex, say v_1, to the others.

In the latter case, say those vertices are $o_1 \in \text{co}\{v_1, v_2\}$, $o_2 \in \text{co}\{v_0, v_1\}$, $o_3 \in \text{co}\{v_1, v_3\}$. We assumed above that $\mathcal{O}_\mathcal{S}$ does not pass through any of vertices v_0. Thus, by Lemma 4, we note that the cones of all these three vertices (and hence of any point in $\mathcal{O}_\mathcal{S}$, as the convex combination of o_1, o_2 and o_3 will have v_1 in its expansion in terms of vertices of $\mathcal{S}$) are subsets of $\mathcal{C}(o_1)$. Thus, $\text{cone}(\mathcal{O}_\mathcal{S}) = \mathcal{C}(o_1)$ and hence by Lemma 7 and Lemma 8, the answer to Problem 2 is affirmative if and only if

$$\mathcal{C}(o_1) \cap \mathcal{B} = \text{cone}(\mathcal{O}_\mathcal{S}) \cap \mathcal{B} \neq \mathbf{0}.$$

In the case where $\mathcal{O}_\mathcal{S}$ passes through one of the vertices of $\mathcal{S}$, say without loss of generality that $o_1 = v_1$, $o_2 \in \text{co}\{v_0, v_2\}$ and $o_3 \in \text{co}\{v_2, v_3\}$ (where neither o_2 nor o_3 coincide with any v_i's). Now, as $\dim \mathcal{B} = 2$, we know by Lemma 13 that there exist linearly independent vectors $b_1 \in \mathcal{B} \cap \mathcal{C}(o_1)$ and $b_2 \in \mathcal{B} \cap \mathcal{C}(o_2)$. Now, define $f : \mathcal{O}_\mathcal{S} \to \mathcal{B}$ by $f(x) = Ax$, where $Ao_1 = b_1$, $Ao_2 = b_2$ and $Ao_3 = b_2$. We first note that o_1, o_2 and o_3 are linearly independent. Thus, the above assignment can be accomplished. Next, we note that $\mathcal{C}(o_2) \subseteq \mathcal{C}(o_3)$ by Lemma 4. Thus, the assignments on the vertices of $\mathcal{O}_\mathcal{S}$ satisfy the cone condition.

Now, let us write any $x \in \mathcal{O}_\mathcal{S}$ as $x = \alpha_1 o_1 + \alpha_2 o_2 + \alpha_3 o_3$, where $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and α_i's are nonnegative. We note that if $\alpha_i \neq 0$, $I(x) \supseteq I(o_i)$. Thus, by Lemma 4, $\mathcal{C}(x) \supseteq \mathcal{C}(o_i)$. As $\mathcal{C}(x)$ is convex, then $\mathcal{C}(x) \supseteq \text{co}\{\mathcal{C}(o_i) : \alpha_i \neq 0\}$. On the other hand, $Ax = \alpha_1 Ao_1 + \alpha_2 Ao_2 + \alpha_3 Ao_3$. Thus, as we have proved that for every i, $Ao_i \in \mathcal{C}(o_i)$ Ax is a convex combination of vectors from $\mathcal{C}(o_i)$, where $\alpha_i \neq 0$. We noted above that this means $Ax \in \mathcal{C}(x)$.

Note that the above proof works for any affine function: if it satisfies the cone criteria on the vertices, it will satisfy those criteria on the rest of $\mathcal{O}_\mathcal{S}$ as well.

Finally, we note the following: with the above notation,

$$f(x) = Ax = \alpha_1 Ao_1 + \alpha_2 Ao_2 + \alpha_3 Ao_3 = \alpha_1 b_1 + (\alpha_2 + \alpha_3) b_2.$$

As b_1 and b_2 are linearly independent and α_1 and $\alpha_2 + \alpha_3$ can not both be zero, $f(x) \neq 0$ for any $x \in \mathcal{O}_\mathcal{S}$. We have thus given a constructive solution for Problem 2 in this case.

3.2.2.2 $\mathcal{O}_\mathcal{S}$ is a quadrilateral

Say without loss of generality that $o_1 \in \text{co}\{v_0, v_2\}$, $o_2 \in \text{co}\{v_0, v_3\}$, $o_3 \in \text{co}\{v_1, v_2\}$ and $o_4 \in \text{co}\{v_1, v_3\}$ (where none of o_i's actually coincide with any v_j's). Now, we again know by Lemma 13 that there exist linearly independent vectors $b_1 \in \mathcal{B} \cap \mathcal{C}(o_1)$ and $b_2 \in \mathcal{B} \cap \mathcal{C}(o_2)$.

Now, from the definition of a cone in (1.2), we know that $b_1 \cdot h_3 \leq 0$ and $b_2 \cdot h_2 \leq 0$. We distinguish between two cases: in the first one, without loss of generality, $b_2 \cdot h_2 < 0$.

Since $\{o_1, o_2, o_3\}$ is a linearly independent set, we know that there exist unique coefficients α_i such that

$$o_4 = \sum_{i=1}^{3} \alpha_i o_i.$$

Furthermore, $\alpha_2 > 0$, as $o_4 = \lambda_1 v_1 + \lambda_3 v_3$, with $\lambda_3 > 0$, and $v_3 \notin \text{co}\{o_1, o_3\}$. Now, let us define $f : \mathcal{O}_\mathcal{S} \to \mathcal{B}$ by $f(x) = Ax$, where $Ao_1 = \varepsilon b_1$, $Ao_2 = b_2$, $Ao_3 = \varepsilon b_1$ (we note that $\mathcal{C}(o_1) \subseteq \mathcal{C}(o_3)$ by Lemma 4), and

$$\varepsilon = \frac{-\alpha_2(b_2 \cdot h_2)}{|2(\alpha_1 + \alpha_3)(b_1 \cdot h_2)|}.$$

(If $(\alpha_1 + \alpha_3)(b_1 \cdot h_2) = 0$, let $\varepsilon = 1$.) As $\{o_1, o_2, o_3\}$ is a linearly independent set in $\mathbb{R}^3$, A is well-defined.

Now, we note that, since $o_4 = \alpha_1 o_1 + \alpha_1 o_1 + \alpha_1 o_1$, $Ao_4 = \varepsilon(\alpha_1 + \alpha_3) b_1 + \alpha_2 b_2$ and thus

$$f(o_4) \cdot h_2 = \pm \frac{\alpha_2}{2}(b_2 \cdot h_2) + \alpha_2(b_2 \cdot h_2) \leq \frac{\alpha_2}{2}(b_2 \cdot h_2) < 0.$$

(If $(\alpha_1 + \alpha_3)(b_1 \cdot h_2) = 0$, then $f(o_4) \cdot h_2 = \alpha_2(b_2 \cdot h_2) < 0$.) Thus, the vector assigned to o_4 is in $\mathcal{C}(o_4)$. Thus, f satisfies the cone condition at all four vertices of $\mathcal{O}_\mathcal{S}$. By the proof same as in the case of the triangle, since f is affine, it hence satisfies the cone condition at any point of $\mathcal{O}_\mathcal{S}$.

Finally, we note that for any $x \in \mathcal{O}_\mathcal{S}$, $f(x) = Ax = A(\kappa_1 o_1 + \kappa_2 o_2 + \kappa_3 o_3 + \kappa_4 o_4)$, where $\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 = 1$, and all κ_i's are nonnegative. Thus, $f(x) = (\kappa_1 + \kappa_3 + \kappa_4 \alpha_1 + \kappa_4 \alpha_3)\varepsilon b_1 + (\kappa_2 + \kappa_4 \alpha_2) b_2$. Since b_1 and b_2 are linearly independent, $f(x) = 0$ is thus equivalent to $\kappa_1 + \kappa_3 + \kappa_4 \alpha_1 + \kappa_4 \alpha_3 = 0$ and $\kappa_2 + \kappa_4 \alpha_2 = 0$. Since $\alpha_2 > 0$, and κ_i's are nonnegative, the latter equation implies $\kappa_2 = \kappa_4 = 0$. Hence, the first equation implies $\kappa_1 + \kappa_3 = 0$, which implies $\kappa_1 = \kappa_3 = 0$. As $\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 = 1$, this is clearly impossible. Thus, f is nowhere zero. We are done, having defined a function satisfying Problem 2 on $\mathcal{O}_\mathcal{S}$.

Now, let us assume that $b_1 \cdot h_3 = b_2 \cdot h_2 = 0$. If we remind ourselves that $o_1 \in \text{co}\{v_0, v_2\}$ and $o_2 \in \text{co}\{v_0, v_3\}$, we also know (from the definition of cones at o_1 and o_2) that $b_1 \cdot h_1, b_2 \cdot h_1 \leq 0$. Finally, we can assume that $b_1 \cdot h_2, b_2 \cdot h_3 > 0$. Otherwise, we would have that b_1 or b_2 is in $\text{cone}(\mathcal{O}_\mathcal{S}) \cap \mathcal{B}$, which was solved by Lemma 7.

Now, let us first assume that $b_1 \cdot h_1 < 0$ or $b_2 \cdot h_1 < 0$. Without loss of generality we choose the first option. In that case, let $b_1' = b_1 - cb_2$, where

$$c = \frac{b_1 \cdot h_1}{2(b_2 \cdot h_1)} > 0.$$

(If $b_2 \cdot h_1 = 0$, take $c = 1$ instead.) Now, we note that $b_1' \cdot h_1 = b_1 \cdot h_1 - cb_2 \cdot h_1 = \frac{1}{2} b_1 \cdot h_1 < 0$. (If $b_2 \cdot h_1 = 0$, $b_1' \cdot h_1 = b_1 \cdot h_1 < 0$.) Also, $b_1' \cdot h_3 = b_1 \cdot h_3 - cb_2 \cdot h_3 = -cb_2 \cdot h_3 < 0$. Thus, b_1' is in $\mathcal{C}(o_1)$ and in $\mathcal{B}$ (as it is a linear combination of vectors in $\mathcal{B}$). Furthermore, b_1' and b_2 are still linearly independent and we already established $b_1' \cdot h_3 < 0$. Now, since we have that $b_1' \cdot h_3$ is strictly negative, we can go a few paragraphs back, just using b_1' and b_2 instead of b_1 and b_2 in order to find a constructive answer to Problem 2.

Finally, we have $b_1 \cdot h_1 = 0$ and $b_2 \cdot h_1 = 0$. However, we also know from before that $b_1 \cdot h_3 = b_2 \cdot h_2 = 0$, and that $b_1 \cdot h_2, b_2 \cdot h_3 > 0$. Now, let $b_1' = b_1 - b_2$. Then, $b_1' \cdot h_1 = 0$ and $b_1' \cdot h_3 = b_1 \cdot h_3 - b_2 \cdot h_3 = -b_2 \cdot h_3 < 0$. Thus, again, b_1' is in $\mathcal{C}(o_1)$ and in $\mathcal{B}$, b_1' and b_2 are linearly independent and $b_1' \cdot h_3 < 0$. Again, as before, we can go a few paragraphs back and obtain a solution to Problem 2.

We note that this long and drawn out affair proved the following:

Theorem 16. *Let $n = 3$, $\dim \mathcal{O}_\mathcal{S} = 2$, $\dim \mathcal{B} = 2$ and $v_0 \notin \mathcal{O}_\mathcal{S}$. Assume that $\mathcal{O}_\mathcal{S}$ does not satisfy the conditions of Theorem 15. Then, the answer to Problem 2 is affirmative.*

Thus, we've answered the last remaining case.

References

[1] C. Belta and L. C. G. J. M. Habets, Controlling a class of nonlinear systems on rectangles, *IEEE Transactions on Automatic Control*, Vol. 51, no. 11, pp. 1749–1759, 2006.

[2] C. Belta, L. C. G. J. M. Habets, V. Kumar, Control of multi-affine systems on rectangles with applications to biomolecular networks, *IEEE Conference on Decision and Control*, pp. 534–539, 2002.

[3] M. E. Broucke, Reach control on simplices by continuous state feedback, *SIAM Journal on Control and Optimization*, Vol. 48, no. 5, pp. 3482–3500, 2010.

[4] L. C. G. J. M. Habets and J. H. van Schuppen. Control of piecewise-linear hybrid systems on simplices and rectangles. In *Hybrid Systems: Computation and Control*, ed. by M. D. Di Benedetto and A. L. Sangiovanni-Vincentelli, Springer, 2001, pp. 261–274.

[5] L. C. G. J. M. Habets, P. J. Collins, J. H. van Schuppen, Reachability and control synthesis for piecewise-affine hybrid systems on simplices, *IEEE Transactions on Automatic Control*, Vol. 51, no. 6, pp. 938–948, 2006.

[6] P. C. Kettler, *Vertex and Path Attributes of the Generalized Hypercube with Simplex Cuts*, University of Oslo, 2008.

[7] M. Ornik and M. E. Broucke, On a Topological Obstruction in the Reach Control Problem, *submitted to AMMCS-CAIMS2015 Proceedings*, 2015.

[8] M. Ornik and M. E. Broucke, A topological obstruction to reach control by continuous state feedback, *submitted to IEEE Conference on Decision and Control*, 2015.

[9] B. Roszak and M. E. Broucke, Necessary and sufficient conditions for reachability on a simplex, *Automatica*, Vol. 42, no. 11, pp. 1913–1918, 2006.

[10] E. Semsar-Kazerooni and M. E. Broucke, Reach controllability of single input affine systems, *IEEE Transactions on Automatic Control*, Vol. 59, no. 3, 738–744, 2014.

[11] D. Talman, Intersection theorems on the unit simplex and the simplotope, in *Imperfections and Behavior in Economic Organizations*. Springer, 1994, pp. 257–278.

[12] M. Vukosavljev and M. E. Broucke, Control of a gantry crane: a reach control approach, *IEEE Conference on Decision and Control*, pp. 3609–3614, 2014.