An Obstruction to Solvability of the Reach Control Problem Using Affine Feedback *

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Abstract

This paper studies the reach control problem (RCP) using affine feedback on simplices. The contributions of this paper are threefold. First, we identify a new obstruction to solvability of the RCP using affine feedback and provide necessary and sufficient conditions for occurrence of such an obstruction. Second, for two-input systems, these conditions are formulated in terms of scalar linear inequalities. Third, computationally efficient necessary conditions are proposed for checking the obstruction for multi-input systems as feasibility programs in terms of linear inequalities. In contrast to the previous work in the literature, no assumption is imposed on the set of possible equilibria, so the results are applicable to the general RCP.

Key words: Reach control problem, Affine feedback, Dual cones, Cone conditions

1 Introduction

This paper studies the reach control problem (RCP) using affine feedback on simplices. Given an affine system defined on a simplex $S$, the objective in the RCP is to design a feedback controller such that the trajectories of the closed-loop system leave $S$ in finite time through a prespecified facet, without first leaving it through other facets. The RCP has been the subject of a great deal of research due to its fundamental importance in controlling a subclass of hybrid systems known as piecewise affine systems (Bemporad et al. (2000); Rodrigues (2004); Habets et al. (2006)). Piecewise affine systems are state-based switched systems where each discrete mode has a corresponding continuous-time affine dynamics. The discrete modes correspond to polytopes in the state space. For piecewise affine systems, reach control is at each mode to design a controller that prevents transitions of the closed-loop system to undesired discrete modes, and guarantees transition to the prespecified desired mode. The RCP has found applications in different fields including biomolecular networks (Belta et al. (2002)), robot motion planning (Belta et al. (2005)), aircraft control (Belta and Habets (2006)), robotic manipulators (Martino and Broucke (2014)), and aggressive maneuvers of mechanical systems (Vukosavljev and Broucke (2014)).

The first approaches to solve the RCP in Habets and van Schuppen (2004); Roszak and Broucke (2006) lead to either conservative sufficient conditions or bilinear inequalities that are NP-hard. It later became evident that the (polytopic) set of possible closed-loop equilibria in the simplex, $O_S$, plays a crucial role in solvability of the RCP. In particular, several computationally efficient controller synthesis methods were devised by imposing the assumption that $O_S$ is a face of $S$ (Broucke (2010); Ashford and Broucke (2013); Broucke and Ganness (2014)). The results were extended to polytopes in (Lin and Broucke (2011); Helwa and Broucke (2013)). In Lin and Broucke (2011) the problem was to find a triangulation of the polytope and an associated piecewise affine feedback to solve the RCP assuming the system has $n - 1$ inputs. The goal of (Helwa and Broucke (2013)) was to extend the results of (Broucke (2010)) directly to polytopes by formulating the so-called monotonic RCP. While all these works regard the RCP, the specific problems solved and the approaches are very different from those of this paper. In this paper we focus on a sub-problem of the RCP regarding the ability to assign a non-vanishing affine function on $O_S$. We use a numerical, optimization-inspired approach whereas the previous works exploit system structure to arrive at analytical conditions for solvability. Finally, a Lyapunov theory for the RCP based on so-called flow functions (the analog of Lyapunov functions for stability analysis) was presented in Helwa and Broucke (2015). In this work, we do not assume existence of a flow function, and we are interested in necessary conditions for
solvability rather than the analysis of a given controller.

More closely related to this work, recent research has focused on the existence and structure of the equilibria in the RCP (Semsar-Kazerooni and Broucke (2014)). The notion of reach controllability was introduced to characterize when closed-loop equilibria could be pushed off the simplex using affine feedback. Notions of topological and affine obstructions to solvability arose as necessary conditions to the solvability of RCP (Ornik and Broucke (2015)). The term “obstruction” is used in a similar spirit as in homotopy theory - to extend a continuous (or affine) map on a simplicial complex. The affine obstruction was studied in (Ornik and Broucke (2015)) for the case of two- and three-dimensional systems.

The main contributions of this paper are threefold. First, we formulate necessary and sufficient conditions for existence of a non-vanishing affine extension on $O_S$. To the best of our knowledge, this is the first result in the literature to present an obstruction to solvability of the RCP using affine feedback for multi-input systems and for the most general form of $O_S$. Second, we propose graphically motivated and computationally efficient necessary and sufficient conditions for checking the obstruction on $O_S$ for two-input systems in terms of scalar linear inequalities. Finally, computationally efficient necessary conditions are proposed for checking the obstruction on $O_S$ for multi-input systems as feasibility programs in terms of linear inequalities.

2 Problem Formulation

Consider an $n$-dimensional simplex $S := \text{co}\{v_0, \ldots, v_n\}$, where $v_0, \ldots, v_n$ are $n + 1$ affinely independent points in $\mathbb{R}^n$. Without loss of generality (w.l.o.g.) we assume $v_0 = 0$. Define $S := \{v_0, \ldots, v_n\}$ to be the vertex set of $S$. Let $F_0, \ldots, F_n$ denote the facets of $S$, where each facet is indexed by the vertex it does not contain. We call $F_0$ the exit facet. Let $h_i, j \in \{0, \ldots, n\}$, be the unit normal vector of facet $F_j$ pointing outside of the simplex. Let $0$ denote the singleton set $\{0\}$. Define $I := \{1, \ldots, n\}$ and let $I(x)$ be the minimal index set among $\{0, \ldots, n\}$ such that $x \in \text{co}\{v_i \mid i \in I(x)\}$.

We consider an affine control system on $S$ defined as

$$\dot{x} = Ax + Bu + a, \quad x \in S,$$

where $A \in \mathbb{R}^{n \times n}$, $a \in \mathbb{R}^n$, $B \in \mathbb{R}^{n \times m}$, and rank($B$) $= m$. Define $O := \text{Im}(B)$, the image of $B$. Let $\phi_0(t, x_0)$ denote the trajectory of (1) starting at $x_0 \in S$, under control input $u$, and evaluated at time instant $t$. Reach control theory studies the reachability of the exit facet $F_0$ from any initial point in $S$.

Reach Control Problem (RCP). Consider the affine system (1) defined on a simplex $S$. Find an affine feedback $u(x) := Kx + g$, where $K \in \mathbb{R}^{m \times n}$ and $g \in \mathbb{R}^m$, such that for each $x_0 \in S$ there exist $T \geq 0$ and $\delta > 0$ such that

$$\phi_0(t, x_0) \in S, \forall t \geq 0.$$
Assume that there exists a vertex map \( f \) on \( Q \). The proof is omitted due to its similarity to Theorem 6 in (Roszak and Broucke (2006)).

Based on Lemma 1, if Problem 2 is infeasible then an obstruction to solvability of the RCP using affine feedback exists. For single-input systems, i.e. \( m = 1 \), an immediate consequence of Theorem 1 in Semsar-Kazerooni and Broucke (2014) is that Problem 2 is solvable if and only if \( B \cap \text{cone}(O_S) \neq \emptyset \), where

\[
\text{cone}(O_S) := \bigcap_{i=1}^{\kappa} C(o_i). \tag{6}
\]

However, this only becomes a sufficient condition in the case of multi-input systems (see Example 13). In this paper, we focus on solvability of Problem 2 for systems with more than one input. Necessary and sufficient conditions for the obstruction identified in Problem 2 are formulated in the next section.

3 Necessary and Sufficient Conditions for Existence of a Non-vanishing Affine Extension on \( O_S \)

Let \( B = QR \) be the QR factorization of \( B \). Since \( B \) has full column rank, the columns of \( Q \in \mathbb{R}^{n \times m} \) form an orthonormal basis for \( B \) (Golub and Van Loan (1996), p. 223), such that \( Q^TQ = I_{m \times m} \) and \( QQ^Tb = b \) for all \( b \in B \). Our objective is to find a vertex map \( f \) such that \( f(o_i) \in B \cap C(o_i) \), \( i \in \{1, \ldots, \kappa\} \). To this end, define the convex cone \( C_i \), \( i \in \{1, \ldots, \kappa\} \), as

\[
C_i := \{ w \in \mathbb{R}^m | (Q^T h_j) \cdot w \leq 0, \ j \in I \setminus I(o_i) \} \tag{7}
\]

\[
= \bigcap_{j \in I \setminus I(o_i)} H_j, \tag{8}
\]

where \( H_j \) is a closed convex cone defined as

\[
H_j := \{ w \in \mathbb{R}^m | (Q^T h_j) \cdot w \leq 0 \}, \ j \in I. \tag{9}
\]

Cones \( C_i \) are analogues of the cones \( C(o_i) \) in the original problem set-up. In fact, as \( B \) is mapped by into \( \mathbb{R}^m \) by \( Q^T \), the same mapping transforms \( C(o_i) \cap B \) into \( C_i \). Cones \( H_j \) identify each of the possible constraints impacting \( C_i \).

Lemma 3 Let \( y \in B \). Then \( Q^T y \in C_i \) if and only if \( y \in B \cap C(o_i) \).

PROOF. (\( \Rightarrow \)) Since \( y \in B \) we can write \( QQ^Ty = y \). Assume \( Q^T y \in C_i \). For \( j \in I \setminus I(o_i) \), (7) yields \( (Q^T h_j) \cdot (Q^T y) \leq 0 \) if and only if \( h_j^T QQ^T y \leq 0 \), which implies \( y \in H_j \). The result follows from arguments similar to the first part of the proof.

If \( Q^T h_j \neq 0 \) then \( H_j \) represents a closed half space in \( \mathbb{R}^m \).
If $Q^T h_j = 0$ then $H_j = \mathbb{R}^m$. Since $C_i$ is the intersection of a finite number of half spaces it is a convex polyhedral cone, and it can be written as the convex hull of a finite set of rays. For each $i \in \{1, \ldots, \kappa\}$, let the unit vectors corresponding to these rays be $v_{i,r}, r \in \{1, \ldots, r_i\}$. Therefore, any vector $w \in C_i$ can be written as the conic combination of $v_{i,r}, r \in \{1, \ldots, r_i\}$, i.e., $w = \sum_{r=1}^{r_i} \lambda_r v_{i,r}$, for some $\lambda_r \geq 0$. Next, for $i \in \{1, \ldots, \kappa\}$, define
\[
N_i := \bigcup_{r=1}^{r_i} \{ h \in \mathbb{R}^m \mid h \cdot v_{i,r} < 0 \}. \tag{10}
\]
Each $N_i$ represents the set of all vectors $h \in \mathbb{R}^m$ for which there exists a vector $w \in C_i$ such that $h \cdot w < 0$. In other words, each $N_i$ contains all the vectors $h$ for which there exists an element $w$ which is contained in $C_i$ and in the half-space opposite from $h$. This definition may be inelegant; however, we will soon show that it can be naturally interpreted in relation to the standard notion of dual cones. Note that $N_i$ is a (not necessarily convex) blunt cone (i.e., a cone that does not contain 0) because $h \in N_i$ implies $\lambda h \in N_i$ for any $\lambda > 0$, and $0 \notin N_i$. Theorem 4 formulates a necessary and sufficient condition for the obstruction described by Problem 2 in terms of the intersection of $N_i, i \in \{1, \ldots, \kappa\}$.

Theorem 4 There exists a solution to Problem 2 if and only if
\[
\bigcap_{i=1}^{\kappa} N_i \neq \emptyset. \tag{11}
\]

PROOF. ($\Rightarrow$) Suppose Problem 2 is solvable, i.e., there exist $f(o_i) \in B \cap C(o_i), i \in \{1, \ldots, \kappa\}$, and $\xi \in \mathbb{R}^n$ such that $\xi \cdot f(o_i) < 0$ for all $i \in \{1, \ldots, \kappa\}$. By Lemma 3, $Q^T f(o_i) \in C_i$ and we can write $Q^T f(o_i) = \sum_{r=1}^{r_i} \lambda_r v_{i,r}$, for some $\lambda_r \geq 0$. Since $f(o_i) \in B$ we have $Q^T f(o_i) = f(o_i)$. Therefore, for $i \in \{1, \ldots, \kappa\}$, $\xi \cdot f(o_i) = \xi \cdot Q^T f(o_i) = Q^T \xi \cdot \left( \sum_{r=1}^{r_i} \lambda_r v_{i,r} \right) < 0$. Since $\lambda_r \geq 0$ for all $r \in \{1, \ldots, r_i\}$, there exists $\xi \in \mathbb{R}^n$ such that $(Q^T \xi) \cdot v_{i,r} < 0$. Therefore, $Q^T \xi \in N_i$, for all $i \in \{1, \ldots, \kappa\}$, and (11) is satisfied.

($\Leftarrow$) Suppose (11) holds. Since $N_i$ does not contain 0, there exists a non-zero vector $h^* \in \mathbb{R}^m$ such that for each $i \in \{1, \ldots, \kappa\}$ there exists a vector $v_{i,r}, r \in \{1, \ldots, r_i\}$ such that $h^* \cdot v_{i,r} < 0$. Let $f(o_i) := Qv_{i,r}, i \in \{1, \ldots, \kappa\}$. Clearly, $f(o_i) \in B = \text{Im}(Q)$. Since $Q^T f(o_i) = Q^T Qv_{i,r} = v_{i,r} \in C_i$, by Lemma 3, $f(o_i) \in B \cap C(o_i)$. Next, let $\xi = Qh^*$. Then for $i \in \{1, \ldots, \kappa\}$, $\xi \cdot f(o_i) = \langle Qh^* \rangle \cdot (Qv_{i,r}) = h^* \cdot Qv_{i,r} = h^* \cdot v_{i,r} < 0$. □

To the best of our knowledge, Theorem 4 is the first result in the literature that provides a necessary and sufficient condition for solvability of Problem 2 for multi-input systems and for the most general form of the set $Q_2$. However, since the cones $N_i$ are not necessarily convex, condition (11) in Theorem 4 leads to a non-convex feasibility problem. Nonetheless, assuming that (11) is satisfied and a vector $h^* \in \bigcap_{i=1}^{\kappa} N_i$ is known, Problem 2 can be formulated as a computationally efficient feasibility program in terms of linear inequalities, as shown in the next corollary.

Corollary 5 Suppose there exists a vector $h^* \in \bigcap_{i=1}^{\kappa} N_i$. The vertex map $f(o_i) = Qw_i, i \in \{1, \ldots, \kappa\}$, is a solution to Problem 2, where $w_i$ always exists and is a solution of the following feasibility program:
\[
\begin{align*}
\text{find} & \quad w_i \in C_i \\
\text{subject to} & \quad h^* \cdot w_i < 0. \tag{12}
\end{align*}
\]

PROOF. Suppose $h^* \in \bigcap_{i=1}^{\kappa} N_i$. Considering (10), for each $i \in \{1, \ldots, \kappa\}$, there exists $r_i \in \{1, \ldots, r_i\}$ such that $h^* \cdot v_{i,r_i} < 0$. Therefore, optimization program (12) is always feasible since clearly $v_{i,r_i} \in C_i$. Since $Q^T Qw_i = w_i \in C_i$, by Lemma 3, $f(o_i) = Qw_i \in B \cap C(o_i)$. For $i \in \{1, \ldots, \kappa\}$, observe that $h^* \cdot w_i = h^* \cdot Qw_i = \langle Qh^* \rangle \cdot (Qw_i) < 0$. Define $\xi := Qh^* \in \mathbb{R}^n$. Therefore, $\xi \cdot f(o_i) < 0$, which completes the proof. □

The necessary and sufficient condition in Theorem 4 is based on the cones $N_i$, which are not necessarily convex. Instead, it is appealing to reformulate the necessary and sufficient condition (11) in terms of the more standard notion of the dual cones, which are convex. This new dual formulation simplifies the presentation of the results in the following sections of the paper. To this end, note that the dual cone of a cone $C \in \mathbb{R}^n$ is defined as (Boyd and Vandenberghe (2004))
\[
C^* := \{ y \in \mathbb{R}^n \mid y \cdot c \geq 0, \quad \forall c \in C \}. \tag{13}
\]

Since every vector in $C_i$ is a conic combination of the vectors $v_{i,r}, r \in \{1, \ldots, r_i\}$, the dual cone of $C_i, i \in \{1, \ldots, \kappa\}$, is written as
\[
C_i^* := \{ y \in \mathbb{R}^m \mid y \cdot v_{i,r} \geq 0, \quad \forall r = 1, \ldots, r_i \}. \tag{14}
\]

Corollary 6 There exists a solution to Problem 2 if and only if
\[
\bigcup_{i=1}^{\kappa} C_i^* \neq \mathbb{R}^m. \tag{15}
\]

PROOF. Considering (10) and (14), it is easy to see that $N_i = \mathbb{R}^m \setminus C_i^*$. Therefore,
\[
\bigcap_{i=1}^{\kappa} N_i \cap (\bigcap_{i=1}^{\kappa} C_i^*) = \mathbb{R}^m \setminus \bigcup_{i=1}^{\kappa} C_i^*. \tag{16}
\]
Thus, conditions (11) and (15) are equivalent. □

Corollary 6 presents necessary and sufficient conditions for the obstruction identified in Problem 2 as a compact and
plausible cone condition. According to Elbassiony and Tiwary (2011), however, determining if the union of a set of polyhedral cones covers $\mathbb{R}^m$ is an NP-complete problem. Therefore, Corollary 6 (and analogously Theorem 4) cannot be efficiently solved using optimization software. In the rest of the paper, we present computationally efficient conditions for solvability of Problem 2. In particular, (i) for two-input systems, necessary and sufficient conditions are presented in Section 4 in terms of easily verifiable convexity relations, and (ii) for general systems, necessary conditions are presented in Section 5 as feasibility programs in terms of linear inequalities. The results of Sections 4 and 5 can be easily programmed and solved using available optimization software.

4 Two-input Systems

This section is focused on two-input systems, i.e., systems for which $B$ is a 2-dimensional sub-space of $\mathbb{R}^n$. This allows us to present a graphical representation of Corollary 6, and propose computationally efficient necessary and sufficient conditions for solving Problem 2 for two-input systems. To this end, consider (8) where $C_i$ is defined as the intersection of a set of closed convex cones $H_j$. According to Theorem 2 in Sundgren (1954), for $i \in \{1, \ldots, \kappa\}$ we can write

$$C_i^* = \mathcal{co}\left\{ \bigcup_{j \in I \setminus I(o_i)} H_j^* \right\} = \mathcal{co}\{H_j^* | j \in I \setminus I(o_i)\}, \quad (17)$$

where $H_j^*$ is the dual cone of $H_j$. The set $H_j^*$ is either a ray or the singleton 0, as proved in the following lemma.

Lemma 7 If $Q^T h_j \neq 0$ then $H_j^*$ is a ray given by $H_j := \{\alpha Q^T h_j | \alpha \geq 0\}$. Otherwise, $H_j^* = 0$.

PROOF. Considering (9) and (13), for all $j \in I$, we can write

$$H_j^* = \{y | y \cdot w \geq 0, \forall w \in H_j\} \quad (18)$$

$$= \{y | y \cdot w \geq 0, \forall w \text{ s.t. } -(Q^T h_j) \cdot w \geq 0\}.$$ 

If $Q^T h_j = 0$ then $H_j^* = \{y | y \cdot w \geq 0, \forall w \in \mathbb{R}^m\} = 0$. Next, assume $Q^T h_j \neq 0$. Any vector $\alpha y^*$, where $y^* = -Q^T h_j$, lies in $H_j^*$ if and only if $\alpha \geq 0$. We claim that any vector $y$ that is not collinear with $y^*$ does not lie in $H_j^*$. W.l.o.g, let $y := \beta y^* + \beta_1 y_1^*$, where $y^* \cdot y_1^* = 0$, $y_1^* \neq 0$, $\beta, \beta_1 \in \mathbb{R}$, and $\beta_1 \neq 0$. Equation (9) yields $y_1^* \in H_j$ and $-y_1^* \in H_j$. However, $y \cdot y_1^* = \beta |y_1^*|^2$ and $y \cdot (-y_1^*) = -\beta \cdot |y_1^*|^2$. Therefore, either $y \cdot y_1^*$ or $y \cdot (-y_1^*)$ is less than zero. Hence, by (18), $y \notin H_j^*$. □

W.l.o.g., by reordering indices, assume $Q^T h_j \neq 0$, $j \in \{1, \ldots, n\}$, where $0 \leq n' \leq n$. Now assume the system has two inputs, i.e., assume $m = 2$. W.l.o.g., by reordering indices $j \in \{2, \ldots, n\}$, assume the rays $H_j^*, j \in \{1, \ldots, n\}$, are arranged in clockwise order (see Fig. 4 for an example). Let $H_0^* := H_0^*$ and define $H_j := \{y \mid -y \in H_j^*\}, j \in \{0, \ldots, n\}$. The following lemma presents three special cases where the obstruction described by Problem 2 does not exist.

Lemma 8 Suppose the affine system has two inputs. Problem 2 is solvable if one of the following conditions is satisfied.

(i) $n' \leq 2$

(ii) $\exists j^* \in \{0, \ldots, n' - 1\}$ such that $H_{j^*+1}^* = -H_{j^*}^*$.

(iii) $\exists j^* \in \{0, \ldots, n' - 1\}$ such that $H_j^* \subseteq \mathcal{co}\{H_{j^*}^*, H_{j^*+1}^*\}, j \in \{1, \ldots, n\}$

PROOF. (i) Assume $n' \leq 2$. Considering (17), we have $C_i^* \subseteq \mathcal{co}\{0 \cup \left(\bigcup_{1 \leq j \leq n'} H_j^*\right)\}, i \in \{1, \ldots, \kappa\}$. Hence, $\bigcup_{i=1}^\kappa C_i^* \subseteq \mathcal{co}\{0 \cup \left(\bigcup_{1 \leq j \leq n'} H_j^*\right)\}$. Considering Fig. 3, the set $\mathcal{co}\{0 \cup \left(\bigcup_{1 \leq j \leq n'} H_j^*\right)\}$ is equal to the singleton 0 if $n' = 0$, the ray $H_j^*$ if $n' = 1$, or the cone $\mathcal{co}\{H_j^*, \hat{H}_{j+1}^*\}$ if $n' = 2$. Therefore, there always exists a vector $h^* \notin \mathcal{co}\{0 \cup \left(\bigcup_{1 \leq j \leq n'} H_j^*\right)\}$ and, by Corollary 6, Problem 2 is solvable.

(ii) Assume there exists $j^* \in \{0, \ldots, n' - 1\}$ such that $H_{j^*+1}^* = -H_{j^*}^*$. Since the rays are arranged in clockwise order, all rays $H_j^*, j \in \{1, \ldots, n\}$, lie in the same side of the line $H_{j^*}^* \cdot H_{j^*+1}^*$, in a closed half-plane $\mathcal{P}$ (see Fig. 4). Considering (17), the cones $C_i^*, i \in \{1, \ldots, \kappa\}$, are subsets of $\mathcal{P}$. Hence by Corollary 6, Problem 2 is solvable.

(iii) Assume there exists $j^* \in \{0, \ldots, n' - 1\}$ such that $H_{j^*+1}^* \subseteq \mathcal{co}\{H_{j^*}^*, \hat{H}_{j^*+1}^*\}, \forall j \in \{1, \ldots, n\}$. Hence, considering Lemma 7, each vector $-Q^T h_j, j \in \{1, \ldots, n\}$, can be written as a conic combination of the two vectors $-Q^T h_j$ and $-Q^T h_{j+1}^*$. Recall that the indices were reordered w.l.o.g. such that $Q^T h_j = 0, j \in \{n' + 1, \ldots, n\}$. Furthermore, according to (17) and Lemma 7, any vector in $C_j^*$ can be written as a conic combination of the vectors $-Q^T h_j, j \in I \setminus I(o_i)$. Therefore, any vector in $C_j^*$ can be written as a conic combination of the two vectors $-Q^T h_j$. □
and considering the fact that \( i \in \{1, \ldots, \kappa\} \), is either 0 or any of the rays \( H_j^* \) or \( H_j^* +1 \). Hence, there exists a vector \( h^* \in \text{co}\{H_j^*, H_j^* +1\} \) such that \( h^* \notin C_i^*, i \in \{1, \ldots, \kappa\} \), which is in contradiction with \( \bigcup_{i=1}^{\kappa} C_i^* = \mathbb{R}^2 \).

\[ \square \]

The results of the following Lemma are used to address the case where none of the conditions in Lemma 8 is satisfied.

**Lemma 9** The following statements hold for two-input systems.

(i) The dual cones \( C_i^* \), \( i \in \{1, \ldots, \kappa\} \), are convex and their boundaries are among \( H_j^* \), \( j \in I \).

(ii) If \( \bigcup_{j=0}^{n'} \text{co}\{H_j^*, H_j^* +1\} = \mathbb{R}^2 \), then \( \bigcup_{i=1}^{\kappa} C_i^* = \mathbb{R}^2 \) if and only if each \( \text{co}\{H_j^*, H_j^* +1\}, j \in \{0, \ldots, n'-1\} \), is contained in some \( C_i^* \).

**Proof.** (i) This can be verified by (17) and Theorem 2 in Gerstenhaber (1951).

(ii) \((\Leftarrow)\) Assume each \( \text{co}\{H_j^*, H_j^* +1\}, j \in \{0, \ldots, n'-1\} \), is contained in some \( C_i^* \). Then, using the assumption in part (ii) of the lemma, \( \mathbb{R}^2 = \bigcup_{j=0}^{n'-1} \text{co}\{H_j^*, H_j^* +1\} \subseteq \bigcup_{i=1}^{\kappa} C_i^* \), i.e. \( \bigcup_{i=1}^{\kappa} C_i^* = \mathbb{R}^2 \).

\((\Rightarrow)\) Let \( \bigcup_{i=1}^{\kappa} C_i^* = \mathbb{R}^2 \). Now assume there exists \( j^* \in \{0, \ldots, n'-1\} \) such that \( \text{co}\{H_j^*, H_j^* +1\} \not\subseteq C_i^*, i \in \{1, \ldots, \kappa\} \). Since the boundaries of \( C_i^* \) are among \( H_j^* \) (part (i) of the lemma) and considering the fact that \( H_j^* \) and \( H_j^* +1 \) are in consecutive (clockwise) order, we can conclude that the intersection of \( \text{co}\{H_j^*, H_j^* +1\} \) with \( C_i^* \), \( i \in \{1, \ldots, \kappa\} \), is \( H_j^* \) or \( H_j^* +1 \). Based on Lemma 9, if the union of convex hulls \( \bigcup_{j=0}^{n'-1} \text{co}\{H_j^*, H_j^* +1\} \) covers the whole space, then \( \bigcup_{i=1}^{\kappa} C_i^* = \mathbb{R}^2 \) and only if each convex hull \( \text{co}\{H_j^*, H_j^* +1\} \) is contained in some \( C_i^* \). Furthermore, according to Corollary 6, Problem 2 is infeasible if and only if the union of the cones \( C_i^* \) covers \( \mathbb{R}^2 \). In other words, if \( \bigcup_{j=0}^{n'-1} \text{co}\{H_j^*, H_j^* +1\} = \mathbb{R}^2 \), Problem 2 is infeasible if and only if each pair of rays \( H_j^* \) and \( H_j^* +1 \) is contained in some \( C_i^* \). Thus, we need to know each ray \( H_j^*, j \in \{1, \ldots, n'\} \), is contained in which cones \( C_i^*, i \in \{1, \ldots, \kappa\} \). To this end, for \( j \in \{1, \ldots, n'\} \), define

\[ J_j := \{ i \in \{1, \ldots, \kappa\} \mid j \in I \setminus I(o_1) \}. \]  

(19)

Considering (17), if \( i \in J_j \) then \( H_j^* \subseteq C_i^* \). However, there may exist \( j^* \) and \( i^* \) such that \( H_j^* \subseteq C_i^* \) but \( i^* \notin J_j \). For example, assume \( I = \{1, 2, 3, 4\} \) and \( I(o_1) = \{0, 4\} \). Let the rays \( H_3^*, H_2^* \), and \( H_1^* \) be as shown in Fig. 6. Here, \( H_4^* \subseteq C_4^* \), but \( 1 \notin J_4 \). Therefore, we should update the sets \( J_j \) by adding new indices \( i^* \) that satisfy \( H_j^* \subseteq C_i^* \). Note that if \( C_j^* \) contains the rays \( H_j^* \) and \( H_j^* +1 \), then by convexity \( C_j^* \) contains any ray \( H_j^* \subseteq \text{co}\{H_j^*, H_j^* +1\} \), where \( j_1, j_2, j_3 \in \{1, \ldots, n'\} \). Based on this fact, Algorithm 10 updates the sets \( J_j \) and stores them in new sets \( \tilde{J}_j \). The members of each set \( \tilde{J}_j \) are indices \( i \in \{1, \ldots, \kappa\} \) such that \( H_j^* \subseteq C_i^* \).

If \( \tilde{J}_j \cap \tilde{J}_{j+1} \neq \emptyset \) for all \( j \in \{0, \ldots, n'-1\} \), then there exists some \( i^* \) such that \( H_j^* \subseteq C_i^* \) and thus the union of the cones \( C_i^* \) covers \( \mathbb{R}^2 \). The following theorem presents computationally efficient necessary and sufficient conditions for checking the obstruction associated with Problem 2 for two-input systems.

**Theorem 11** Suppose the affine system has two inputs and none of the conditions in Lemma 8 is satisfied. Let the index sets \( J_j \) for all \( j \in \{0, \ldots, n'\} \) be computed as in Algorithm 10. Problem 2 is solvable if and only if there exists \( j^* \in \{0, \ldots, n'-1\} \) such that \( \tilde{J}_{j^*} \cap \tilde{J}_{j^* +1} = \emptyset \).
fied, the rays Combining (22) and (23), however, if any vector the convex hull of two rays H
1
and H
2
Next, for (ii) ing to Corollary 6, Problem 2 is solvable. Suppose not. Let H
j
\neq H
j
∗
. Since H
j
∗ = H
j
∗+1 = co{H
j
1
, H
j
2
}, Algorithm 10 guarantees that J
j
is added to J
j
−1 and J
j
+1. This yields J
j
∩ J
j
−1 = \emptyset, a contradiction. Therefore, (20) holds. By (19), observe that ∃ i \in \{1, \ldots, \kappa\} s.t. J
(j
−1)
i \in I \setminus I(o
i
), H
j
∗ = H
j
∗+1 = H
j
∗.

Next, for l1, l2 \in \{1, \ldots, n\}, we claim H
l
∗ = H
l
∗+1 ∈ co{H
l
1
, H
l
2
} ⇔ J
l
1 \cap J
l
2 = \emptyset. (22)

Suppose not. Then Algorithm 10 guarantees that J
l
1 \cap J
l
2 = \emptyset is added to J
l
−1 and J
l
+1. This yields J
l
∩ J
l
−1 = \emptyset, a contradiction. Therefore, (22) holds. By (19), observe that J
l
1 \cap J
l
2 = \emptyset ⇒ ∃ i \in \{1, \ldots, \kappa\} s.t. l1, l2 \in I \setminus I(o
i
).

(23)
The sets C
i
∗, i \in \{1, \ldots, \kappa\}, are m
n
-dimensional cones, where 0 \leq m \leq 2. By (17), a 1-dimensional cone C
i
∗ is the union of some linearly dependent rays H
j
∗, j \in I \setminus I(o
i
). Since by (21), no index j \in \{1, \ldots, n\} such that H
j
∗ = H
j
∗+1 appears in a set I \setminus I(o
i
), i \in \{1, \ldots, \kappa\}, the ray H
j
∗ = H
j
∗+1 cannot be a subset of a 1-dimensional cone. Furthermore, a 2-dimensional cone C
i
∗ is the convex hull of the union of rays H
j
1
and H
j
2
, j1, j2 \in I \setminus I(o
i
). Combining (22) and (23), however, if H
j
∗ = H
j
∗+1 lies in the convex hull of two rays H
j
1
and H
j
2
, then there is no index i \in \{1, \ldots, \kappa\} such that l1, l2 \in I \setminus I(o
i
). Therefore, the ray H
j
∗ = H
j
∗+1 cannot be a subset of a 2-dimensional cone either. Hence, H
j
∗ = H
j
∗+1 \notin \bigcup_{i=1}^{\kappa} C
i
∗ and, according to Corollary 6, Problem 2 is solvable.

(ii) Assume J
j
∩ J
j
−1 = \emptyset and H
j
∗ \neq H
j
∗+1. Pick any vector h∗ in the relative interior of co{H
j
1
, H
j
2
, H
j
∗+1} (note that since condition (ii) in Lemma 8 is not satisfied, the rays H
j
1
and H
j
2
are linearly independent and the relative interior is non-empty). We prove by contradiction that h∗ \notin \bigcup_{i=1}^{\kappa} C
i
∗. Assume h∗ ∈ C
i
∗ for some

Theorem 11. Theorem 11.

Now, suppose Problem 2 is solvable. According to Corollary 6, \bigcup_{i=1}^{\kappa} C
i
∗ \neq \mathbb{R}^m. Suppose by way of contradiction J
j
∩ J
j
−1 \neq \emptyset for all j \in \{0, \ldots, n
n
−1\}. Therefore, for each j \in \{0, \ldots, n
n
−1\}, there exists j∗ ∈ \{1, \ldots, \kappa\} such that j, j+1 \in I \setminus I(o
i
). By (17), co{H
j
∗, H
j
∗+1} ⊆ C
j
∗. Therefore, \bigcup_{j=0}^{n
n
−1} co{H
j
∗, H
j
∗+1} \subseteq \bigcup_{i=1}^{\kappa} C
i
∗, which is a contradiction because \bigcup_{i=1}^{\kappa} C
i
∗ \neq \mathbb{R}^m and we showed in the previous argument that \bigcup_{j=0}^{n
n
−1} co{H
j
∗, H
j
∗+1} = \mathbb{R}^m. \square

Remark 12 (Computational complexity) It was assumed that the rays H
j
∗, j \in \{1, \ldots, n\}, are arranged in clockwise order. The time complexity of efficient sorting algorithms is known to be in O(n \log n) Knuth (1998). Lemma 8 requires verification of three conditions: condition (i) is trivially checked in constant time. Conditions (ii) and (iii) require at most O(n) and O(n^2) operations, each of which can be formulated as scalar linear equalities or inequalities. Algorithm 10 contains, at worst, O(n^3) repetitions of a set union and intersection operation. Thus, the total complexity of Algorithm 10 may vary depending on the data structure used in the implementation, but it can certainly be bounded.
by $O(n^4)$. Theorem 11 has the same computational complexity.

Next, we use the results of this section to solve Problem 2 for an affine system with two inputs. Example 13, which is four-dimensional as by Ornik and Broucke (2015) lower-dimensional examples will not be particularly illustrative, is important for two reasons. First, it shows that the necessary condition proposed in Semsar-Kazerouni and Broucke (2014) for single-input systems, i.e., $B \cap \text{cone}(O_S) \neq \emptyset$, where cone$(O_S)$ is defined in (6), is not a necessary condition for solvability of Problem 2 for multi-input systems. Second, as mentioned earlier, since $O_S$ is not a face of $S$, previous results in the literature cannot address the solvability of the RCP in Example 13, without resorting to some triangulation of $S$.

Example 13 Consider a simplex $S = \text{co}\{v_0, \ldots, v_4\} \subseteq \mathbb{R}^4$, where $v_i = [0 \ 0 \ 0 \ 0]$, and for $j \in I := \{1, 2, 3, 4\}$, define $v_j = e_j$, the $j^{th}$ Euclidean basis vector. Therefore, $h_i := -e_j$, $j \in I$. Let the parameters of the affine system (1) be

$$A = \begin{bmatrix} -8 & -8 & 2 & -6 \\ 4 & 16 & 0 & 4 \\ 4 & 10 & 0 & 4 \\ -4 & 2 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & -1 \\ 3 & 2 \\ 2 & 1 \\ 0 & -1 \end{bmatrix}, \quad a = [0 \ 1 \ 0 \ 0].$$

Based on (4), it is easy to verify that

$$O = \left\{ x \in \mathbb{R}^4 \mid -8 \leq 16 \leq 0 \leq 0 \leq -2 \leq -2 \right\},$$

and $O_S = \text{co}\{o_1, o_2, o_3\}$, where $o_1 = (0, 0, 0, 1/2)$, $o_2 = (1/2, 0, 0, 0)$, and $o_3 = (1/2, 1/2, 0, 0)$. Note that $I\{o_1\} = \{0, 1\}$, $I\{o_2\} = \{0, 1, 2\}$, and $I\{o_3\} = \{0, 1, 2\}$. The QR factorization of $B$ is computed as

$$Q = \begin{bmatrix} 0.64 & -0.64 & -0.43 & 0.38 \\ -0.41 & -0.41 & 0.82 & 0.82 \\ -0.41 & -0.41 & 0.82 & 0.82 \\ -0.41 & -0.41 & 0.82 & 0.82 \end{bmatrix}^T, \quad R = \begin{bmatrix} 6.49 & 2.35 \\ 0.00 & 1.22 \end{bmatrix}.$$ 

The cones $C_i$, $i \in \{1, 2, 3\}$, are shown in Fig. 8(a) and it is easy to see that their intersection is the singleton $0$. Using Lemma 3, we conclude that $B \cap \text{cone}(O_S) = \emptyset$. Next, we use Theorem 11 to solve Problem 2. This will show that the necessary condition proposed in Semsar-Kazerouni and Broucke (2014) for single-input systems, i.e., $B \cap \text{cone}(O_S) \neq \emptyset$, is no longer necessary for two-input systems. Note that $Q^T h_j \neq 0$ for all $j \in I$ and the rays $H_j^*$, $j \in I$, are already in clockwise order as illustrated in Fig. 8(b). It is easy to see that $J_1 = \{1\}$, $J_2 = \{1, 2\}$, $J_3 = \{1, 3\}$, and $J_4 = \{2, 3\}$. Next, Algorithm 10 yields $J_1 = \{1\}$, $J_2 = \{1, 2\}$, $J_3 = \{1, 2, 3\}$, and $J_4 = \{2, 3\}$. Since $J_1 \cap J_2 = \emptyset$, by Theorem 11, Problem 2 is solvable. This is also in accordance with Corollary 6, since $\bigcup_{i=1}^{3} C_i \neq \mathbb{R}^2$ as shown in Fig. 8(b). Next, we use Corollary 5 to compute a vertex map $f : V_{O_S} \rightarrow B$ that solves Problem 2. Consider the vector $h^* = [1.5 \ 0.42]^T \in \text{co}\{H_1^*, H_2^*, H_3^*\}$.

![Fig. 8](image_url)  

Fig. 8. The graphics corresponding to Example 13. The rays are shown with dashed lines, the vectors are shown with solid arrows, and the cones are illustrate by blue arcs. (a) the cones $C_i$, $i \in \{1, 2, 3\}$, and (b) the corresponding dual cones $C_i^*$, $i \in \{1, 2, 3\}$, $v_{1,1}, v_{1,2}$ are the boundary rays for cones $C_i$.

Clearly, $h^* \notin \bigcup_{i=1}^{3} C_i^*$ and, by (16), $h^* \in \bigcap_{i=1}^{3} \mathcal{N}_i$. Solving (12), we find the following feasible vectors: $w_1 = [-33.6 -120.46]^T \in C_1, w_2 = [-67.71 \ 35.1]^T \in C_2,$ and $w_3 = [-90.59 \ 47.59]^T \in C_3$. The corresponding closed-loop vertex map $f(a) = Q w_i, i \in \{1, 2, 3\},$ is then computed as $f(a_1) = (27.69, 70.67, 14.33, -98.36)$, $f(a_2) = (-57.64, 28.98, 28.87, 28.66)$, and $f(a_3) = (-77.37, 38.51, 38.63, 38.86)$. We can use the vertex map $f$ to find an affine feedback $u(x) = K x + g$ such that the closed-loop vector field is non-vanishing on $O_S$ and satisfies $A x + B u(x) + a \in C(x), x \in O_S$. To this end, the vertex map $f$ is extendible on $O_S$ to the affine map $F = K_F x + g_F$, where

$$K_F = \begin{bmatrix} 123.52 & 2.8 & 1.48 & 55.37 \\ 112.11 & 0.86 & -0.09 & 139.34 \\ 115.21 & 1.36 & 0.14 & 28.66 \\ 113.41 & 6.34 & 0.61 & -196.71 \end{bmatrix}, \quad g_F = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$ 

Note that $K_F = A + BK = A + QR K$ and $g_F = Bg + a = QR g + a$. Hence, the affine feedback controller gains $K = R^{-1} Q^T (K_F - A)$ and $g = R^{-1} Q^T (g_F - a)$ are computed as

$$K = \begin{bmatrix} 114.31 \ -2.15 \ 0.38 \ -86.03 \end{bmatrix}, \quad g = \begin{bmatrix} 0 \ -117.41 \ -4.34 \ -0.61 \ 196.71 \end{bmatrix}.$$ 

It is easy to verify that the closed-loop affine system defined over simplex $S$ satisfies the conditions in Corollary 9 in Roszak and Broucke (2006) with $\xi = [-5.37 \ 5.83 \ 0.87 \ 5.27]^T$. Therefore, the RCP is solvable even though $B \cap \text{cone}(O_S) = \emptyset$.

5 Computationally Efficient Necessary Conditions for Existence of a Non-Vanishing Affine Extension on $O_S$

In this section, we present necessary conditions for solvability of Problem 2 as a feasibility program in terms of linear inequalities. Feasibility problems subject to linear inequalities can easily be programmed and solved using available
optimization software. In contrast to Section 4, the necessary conditions are applicable to all systems regardless of the number of inputs.

Let \( I \) be an \((m + 1)\)-subset of \( \{1, \ldots, \kappa\} \), i.e. \( I \) is a subset of \( \{1, \ldots, \kappa\} \) with \( m + 1 \) elements. For each \( i \in I \), let \( E_I(o_i) \subseteq I(o_i) \) be the set of non-zero exclusive members of \( I(o_i) \) in the set \( \bigcup_{i \in I} I(o_i) \), defined as \( E_I(o_i) := \{ i \in I(o_i) \mid i \not\in I(o_j), \forall j \in I, j \neq i \} \). Next, define \( S_I \) to be the set of non-zero shared vertices in \( \bigcup_{i \in I} I(o_i) \) given by

\[
S_I := \left( \bigcup_{i \in I} I(o_i) \right) \setminus \left( \bigcup_{i \in I} E_I(o_i) \cup \{0\} \right).
\] (24)

The following lemma is used in the proof of the main result of this section.

**Lemma 14** Each ray \( H_j^* \), \( j \in I \setminus S_I \), is contained in all, except (potentially) one, cones \( C_i^* \), \( i \in I \).

**PROOF.** Equation (24) yields \( I \setminus S_I = (I \setminus \bigcup_{i \in I} E_I(o_i)) \cup \bigcup_{i \in I} E_I(o_i) \). First, assume \( j \in I \setminus \bigcup_{i \in I} E_I(o_i) \). Then \( I \setminus \bigcup_{i \in I} E_I(o_i) \subseteq I \setminus I(o_i), i \in I \). (17) yields \( H_j^* \subseteq C_i^*, i \in I \). Therefore, each ray \( H_j^* \), \( j \in I \setminus \bigcup_{i \in I} E_I(o_i) \), is contained in all \( m + 1 \) cones \( C_i^*, i \in I \). Second, assume \( j \in E_I(o_i), i' \in I \). Since

\[
E_I(o_i) \subseteq I \setminus I(o_i), i' \in I \setminus \{i'\},
\] (17) yields \( H_j^* \subseteq C_i^*, i \in I \setminus \{i'\} \). Therefore, each ray \( H_j^* \), \( j \in E_I(o_i), i' \in I \), is contained in all cones \( C_i^*, i \in I \), except potentially \( C_i^* \). This completes the proof. \( \Box \)

Given the set \( I \), let the cone \( M_I \) be defined as

\[
M_I := \{ y \in \mathbb{R}^m \mid h_j \cdot y \leq 0, j \in I \setminus S_I \}. \] (25)

The next result shows that if \( B \) is not of sufficiently high dimension, and there are no non-zero vectors lying in \( B \cap M_I \), then the obstruction described by Problem 2 exists.

**Theorem 15** Assume \( 1 \leq m < \kappa \) and let \( I \) be an \((m + 1)\)-subset of \( \{1, \ldots, \kappa\} \). If Problem 2 is solvable, then \( B \cap M_I \neq 0 \).

**PROOF.** By way of contradiction, assume Problem 2 is solvable, but \( B \cap M_I = 0 \). Define the cone \( M_I \) as

\[
M_I := \{ w \in \mathbb{R}^m \mid (Q^T h_j) \cdot w \leq 0, j \in I \setminus S_I \} = \bigcap_{j \in I \setminus S_I} H_j,
\]

where \( H_j \) is defined in (9). Similar to Lemma 3, it is easy to show that if \( y \in B \), then \( Q^T y \in M_I \) and only if \( y \in B \cap M_I \). By assumption, \( B \cap M_I = 0 \). Therefore, \( M_I = 0 \), and \( M_I^* = \mathbb{R}^m \), where \( M_I^* \) is the dual cone of \( M_I \). Using Theorem 2 in Sandgren (1954), we can write

\[
M_I^* = \bigcap_{j \not\in I \setminus S_I} H_j^* = \text{co} \{ H_j^* \mid j \in I \setminus S_I \}.
\]

Based on Theorem 3.2 in Beck and Robins (2007), the \( m \)-dimensional cone \( M_I^* \) can be triangulated into simplicial cones, where each simplicial cone is defined as the convex hull of precisely \( m \) linearly independent rays \( H_j^* \), \( j \in I \setminus S_I \). Therefore,

\[
M_I^* = \bigcup_{j \not\in I \setminus S_I} \text{co} \{ H_j^* \} = \bigcup_{j \not\in I \setminus S_I} H_j^*.
\]

Since \( M_I^* = \mathbb{R}^m \), we have \( \bigcup_{i=1}^{\kappa} C_i^* = \mathbb{R}^m \). Therefore, according to Corollary 6, Problem 2 is unsolvable, which is a contradiction. \( \Box \)

Based on Theorem 15, if Problem 2 is solvable, the cone condition \( B \cap M_I \neq 0 \) must hold for any \((m + 1)\)-subset of \( \{1, \ldots, \kappa\} \). The number of \((m + 1)\)-subsets of \( \{1, \ldots, \kappa\} \) is given by the binomial coefficient. Therefore, Theorem 15 presents \( \binom{\kappa}{m+1} \) sets of necessary conditions for solvability of Problem 2. These conditions rely on determining an intersection of up to \( n \) half-spaces in \( \mathbb{R}^m \), which was shown in Brown (1978) to be equivalent to the multi-dimensional convex hull problem. This problem can be solved in time polynomial in \( n \). One of the algorithms was given in Skiena (2008), with the worst case complexity of \( O(n^{m/2} + 1) \).

We finish this section with an interesting special case where the familiar condition \( B \cap \text{cone}(O_S) \neq 0 \) (Semsar-Kazerooni and Broucke (2014)) emerges from Theorem 15 as both a necessary and sufficient condition.

**Assumption 16** Any shared vertex index \( j \in I \) is shared by all \( I(o_i), i = 1, \ldots, \kappa \).

**Corollary 17** Suppose Assumption 16 holds and \( 1 \leq m < \kappa \). Problem 2 is solvable if and only if \( B \cap \text{cone}(O_S) \neq 0 \).

\( ^1 \) Since the convex hull of rays \( H_j^*, j \in I \setminus S_I \), covers the whole space \( \mathbb{R}^m \), the cardinality of the set \( I \setminus S_I \) is greater than or equal to \( m \).
Thus, I vertices $S$ for any $0$

Suppose Assumption 16 holds. Then for any set $B$.


