NUMERICAL ESTIMATION OF BIDIRECTIONAL PLANT-CONTROL DESIGN COUPLING IN CONTROL CO-DESIGN

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ABSTRACT

Control co-design has been shown to be an effective tool for optimizing engineering systems. One characteristic feature of these problems is the presence of design coupling between the plant and controller, and this coupling is one of the key deciding factors when selecting a solution strategy for the design optimization problem. In this paper, we provide a method to numerically estimate design coupling strength, accounting for bidirectional coupling (plant and control mutually affect each other), and for estimating design coupling when the control design variables are time-varying trajectories. Design coupling strength is one of several factors that can affect what design optimization problem formulation decisions will best meet the needs of the broader engineering design effort. Some guidelines are provided here for how to use design coupling information to help support problem formulation decisions, including identifying the most impactful design variables and design variable relationships. The coupling strength estimation and its use for problem formulation is illustrated with a spacecraft control co-design (CCD) example. In the example, the plant (physical system) and attitude control design of a satellite are optimized, and design coupling strength information is used to identify the most appropriate plant design variable choice out of a small pool of candidates.

Keywords: Control Co-Design, Design Coupling, Optimal Control, Spacecraft Control, Optimization, Optimization Problem Formulation

1. INTRODUCTION

Control Co-Design (CCD) problems [1, 2] aim to optimize the physical system (plant) and control design of an engineering system holistically. CCD methods have been used to improve the design of engineering systems in many domains, including energy production and storage [3–5], aerospace [6], vehicle suspension [7], and robotics [8]. One important aspect to consider when formulating and solving these types of optimization problems is the presence of design coupling. The presence of design coupling in a CCD problem implies that control design decisions can affect optimal plant design, and vice versa. This coupling can be bidirectional if there is a dependence of the control and design problems on each other, or unidirectional if only the dependence of the optimal control on plant design is considered [9, 10]. If the optimization problems have bidirectional design coupling, the sequential solution of the plant and control problems, that is, first optimizing the plant design and then the control inputs, is not guaranteed to result in a system-optimal solution [2]. Examples of achieving a suboptimal result when solving a coupled problem sequentially are provided in Refs. [2, 11].

There exist different ways to estimate coupling in CCD problems, with most of the coupling measures in the literature having been derived through the optimality conditions. In Refs. [2, 12] the authors define a CCD problem as the weighted sum of two objectives: one for the control design and one for the plant design. In that same work, the combined plant and control optimality conditions are used to study plant and control coupling. The coupling term is obtained as the difference between the KKT conditions for plant optimality and Pontryagin conditions for control optimality, for the separated problems, and the optimality conditions for the CCD problem. This coupling measure has also been used in Refs. [13, 14]. One key assumption in this earlier work is that the coupling is unidirectional, neglecting the effect of the control problem on the plant design problem. This same approach is taken in Ref. [15], where the authors provide a coupling measure under the assumption that the system dynamics are linear and time-invariant. In Ref. [16] it is shown that the coupling term derived in Ref. [2] is equivalent to a different coupling term obtained by comparing the KKT conditions for the CCD problem to the plant optimization KKT conditions [12]. In Ref. [16], the authors use the coupling term to calculate the gradient of the design optimization problem, reducing the computational cost. Additionally, they also demonstrate how the coupling measure can be used to decide if a CCD problem can be solved sequentially instead of simultaneously. The previous work was extended to

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account for bidirectional coupling in Ref. [9], however, the control design variables considered throughout the paper were scalar quantities, and its use was not demonstrated. In this work, we will provide a method for estimating design coupling strength when the control design variables are time-varying quantities.

Another approach to calculating design coupling is described in Ref. [17], where a general method is presented to obtain all the sensitivities, or derivatives, of the optimal objective, design variables, constraints, and Lagrange multipliers with respect to problem parameters of a finite-dimensional optimization problem. These are obtained by differentiating the objective function and the KKT conditions and solving the system of equations. The previous derivations are extended to CCD, with the objective sensitivity calculated for infinite-dimensional problems using calculus of variations. In that same work, the objective sensitivity with respect to a parameter or function is obtained by calculating the gradient of its Lagrangian function with respect to the parameter/function evaluated at the optimal solution. Although it is not a CCD article, in Ref. [18] design coupling is estimated by calculating the derivatives of the optimal value of each design variable with respect to other variables for a finite-dimensional multidisciplinary design optimization (MDO) problem; a similar approach is used in this article.

The methods for estimating design coupling that have been mentioned so far require the solution of the optimization problem, or at least partially solving the KKT conditions, to calculate coupling. A coupling measure that does not require solving the KKT conditions or calculating any partial derivatives is presented in Ref. [19]. In that work, the CCD problem is formulated as in Ref. [2]; the optimization problem has two objective function terms, one for the plant and one for the controller, with the overall system objective being a linear combination of the two. That work also assumes that design coupling is unidirectional. The paper shows that for some problem formulations, the coupling strength can be determined a priori using the controllability Grammian, and it also provides the coupling measure for an infinite-horizon LQR problem.

The information provided by the design coupling measure can be used to support problem formulation and solution implementation decisions. Optimization problem formulation tasks focus on formulating the optimization problem in a way such that it is well-posed and solvable, as well as using models that approximate the system well. The three categories of optimization problem formulation decisions are design representation, comparison metrics, and predictive models [20, 21]. While heuristics are often used in problem formulation, we hope that the use of design coupling will aid in making problem formulation a more systematic task, especially when the optimization user is not able to rely on intuition or previous knowledge. In this work, we will focus on using design coupling quantification to aid design variable selection, which relates primarily to design representation decisions (i.e., what is the design optimization variable space). Design coupling strength is not the only metric to consider when constructing problem formulations, but it is the focus of this work. Other quantitative tools that can help to support problem formulation decisions include problem conditioning and monotonicity analysis [22]. It should be noted that the design variable selection task is not isolated, and is affected by choices made regarding how to compare and objectively rank design candidates, and what models to use for predicting the effects of design changes on comparison metrics (objective and constraint functions). Similarly, the way the optimization problem is formulated will affect what solution strategies are appropriate for a particular problem.

If the objective function and predictive model decisions are fixed, an important decision is selecting what quantities should be a fixed design parameter and what should be a design optimization variable. Problems with a large number of design variables may be more difficult to solve and more computationally expensive, so it will be of interest to identify reduced-dimension sets of design variables that still result in high-value design studies. Design variable selection can be driven by different goals. Namely, one optimization user could be interested in achieving the lowest cost possible, while a different user might seek to select plant variables that allow solving the control and plant optimization separately while achieving overall system optimality. Robustness is another consideration that might drive design variable selection.

In Ref. [23], the author solves a combined structural and control design optimization of two flexible structures, a two-bar truss and a twelve-bar truss, and computes the sensitivity of the control objective and a system response quantity with changes in plant design. These results can be used to eliminate the least sensitive design variables to identify practically feasible values from the theoretical optimum, and to study the effect of changing the design variable and constraint bounds on the system response. With a few exceptions, most of the previous work has been derived under the assumption that the coupling is unidirectional, thus neglecting the effect of control changes on optimal plant design. When the goal is a realistic consideration of plant design, bidirectional design coupling must be considered as control design decisions influence optimal plant design decisions through physical system failure modes [1]. In the cases where coupling was treated as bidirectional, there were still some simplifications, such as the CCD problem being simple enough to have a closed-form solution, or the control variables being treated as scalars. While these are valid assumptions, CCD problems are in general complex enough that closed-form solutions are not available, and in the case of control design, there are important CCD optimization problems where the design variables are time-varying control inputs. It is therefore necessary to identify a coupling measure that accounts for the bidirectional nature of design coupling in general CCD optimization problems and that allows for control variables to be time-varying. This paper is the starting point for an approach for studying design coupling in CCD in a way in which bidirectional coupling and time dependencies are considered when calculating coupling strength. The main contributions of this work are twofold. First, we present a method for numerically estimating bidirectional coupling strength, including when the control variables are time-varying trajectories. Second, we provide some insights into how to use this information for problem formulation.

The work is structured as follows: Sec. 2 describes a numerical method for estimating design coupling. Next, Sec. 3 describes how design coupling strength information can be used to support
problem formulation decisions. Section 4 presents a numerical example where the plant and control design of a spacecraft is optimized and their coupling is estimated. Finally, Sec. 5 provides concluding remarks and lays out future work opportunities.

2. DESIGN COUPLING STRENGTH

In this work, we are interested in the relationship between optimal plant and optimal control design in control co-design optimization. A general control co-design (CCD) problem can be formulated as:

\[
\min_{x_p, x_c} \Psi = \int_{t_0}^{t_f} \mathcal{L} (t, \dot{x}, x_c, x_p) \, dt + \mathcal{M} (\dot{x} (t_0), \dot{x} (t_f), x_c, x_p)
\]

s.t. \( \dot{x} - f (t, x, x_c, x_p) = 0 \),

\[ C (t, \dot{x}, x_c, x_p) \leq 0 \]

\[ \phi (\dot{x} (t_0), \dot{x} (t_f), x_c, x_p) \leq 0, \]

(1)

where \( x_c \) is the control system design variable vector, \( x_p \) is the physical system design variable vector, \( \Psi \) is the cost function, \( \xi \) are the state trajectories, \( C (t, \dot{x}, x_c, x_p) \) are the path constraints, \( \phi (\dot{x} (t_0), \dot{x} (t_f), x_c, x_p) \) are the boundary constraints, and \( f (t, x, x_c, x_p) \) is the function that describes the system dynamics. It is important to remark that while control theory traditionally considers control inputs as time-varying signals, in CCD \( x_c \) can be control input trajectories \( x_c := u (t) \), or scalar quantities, such as controller gains. This distinction between control trajectory and scalar control inputs will be important when discussing design coupling estimation. This work allows for general time-varying control design variables.

Control co-design problems are a complex class of optimization problems that in most cases require the use of numerical methods such as Direct Transcription to solve them [24]. When it comes to estimating the coupling strength between plant and control variables, it may be possible for simple problems to obtain closed-form equations that describe the plant-control coupling using the calculus of variations, as in Ref. [17]. However, for larger systems with more complex dynamics, analytical expressions are in general not feasible. We then take a numerical approach to estimate the coupling strength. If we look at the structure of some of the coupling measures in the literature, they are based on how the objective function changes with plant and control decisions. Here, we estimate how plant and control design decisions affect each other by calculating the derivatives:

\[
\frac{\partial x^*_p (x_p)}{\partial x_p}, \quad (2)
\]

\[
\frac{\partial x^*_p (x_c)}{\partial x_c}, \quad (3)
\]

where \( x^*_p (x_p) \) is the optimal control design for a given plant design, and \( x^*_p (x_c) \) is the optimal plant design for a given control design. We use these derivatives to estimate how making changes in one variable affects the optimal value of another variable. Since we are considering coupling to be bidirectional in general, the optimal control design will depend on plant design and the optimal plant design will be a function of control design. Equation (2) estimates the sensitivity of one optimal control variable to changes in a single plant variable, and Eq. (3) estimates the sensitivity of a single optimal plant design variable to changes in one control variable. These derivatives estimate how the optimal value of a single variable changes when another variable is modified. Here, we are assuming that only one plant/control variable is changing when we calculate the derivatives. A more global estimation of the coupling should estimate the change of the design variable when all the other variables change, however, considering only changes in one variable helps reduce computational expense. When there are multiple control or plant design variables, the derivatives of each control variable must be calculated for each plant variable, and vice versa. Calculating these derivatives when the control variables are time-varying trajectories is a non-trivial task, it requires the calculation of the derivative of a time-varying function with respect to a parameter, and the calculation of the derivative of a parameter with respect to a time-varying function. This section provides a method for dealing with those issues.

Calculating the derivatives (Eqs. (2)-(3)) requires solving two series of optimization problems. First, we convert the plant design variables to fixed problem parameters and solve the optimal control problem below for a series of plant parameter values, thus obtaining \( x^*_p (x_p) \). If there are multiple plant design variables, only one variable is changed at a time, and the rest remain fixed at their nominal value.

\[
\min_{x_c} \Psi = \int_{t_0}^{t_f} \mathcal{L} (t, \dot{x}, x_c, x_p) \, dt + \mathcal{M} (\dot{x} (t_0), \dot{x} (t_f), x_c, x_p)
\]

s.t. \( \dot{x} - f (t, x, x_c, x_p) = 0 \),

\[ C (t, \dot{x}, x_c, x_p) \leq 0 \]

\[ \phi (\dot{x} (t_0), \dot{x} (t_f), x_c, x_p) \leq 0 \]

(4)

The minimization problem in Eqs. (4) is the same as the general CCD problem formulation in Eqs. (1), except the plant is no longer a design variable; instead, it is a problem parameter. To obtain the control-plant coupling strength, the control variables are converted to parameters, and we optimize the plant design for varying control inputs, obtaining \( x^*_p (x_c) \). There are some considerations to keep in mind when conducting the optimization sweeps to estimate coupling. There is a tradeoff between estimate accuracy and computational expense. To obtain accurate results, the user should select a mesh for the optimization sweeps that is sufficiently fine for useful estimates, however, computational expense will increase with mesh refinement. Another issue to keep in mind is local versus global optimality, which can affect the coupling strength estimate. To improve the likelihood of global optimality and to produce consistent sensitivities, a multistart method was used in the example (Sec. 4). For other problems, it might be necessary to use other methods to robustly identify global optima, such as hybrid gradient-free/gradient-based methods. As mentioned before, dealing with time-varying trajectories makes the estimation of design coupling a challenging task, and we do not have yet a framework for estimating optimal plant design for all possible control trajectory perturbations. In this work, we will assume that the nature of the control perturbations is known to simplify the design coupling estimation. Once
As stated above, the proposed transformation of estimating the sensitivity of the control cost to changes in the plant design is not the only way to convert the coupling information to a scalar quantity. Investigating other transformations is part of ongoing work. To estimate the effect that changes in the control have on optimal plant design, we first need to define in what way the control inputs are changing. Ideally, one would explore the effect that all possible control perturbations have on the plant design, however, this analysis requires a non-trivial definition and the use of additional numerical tools and is left for future work. Here, we will assume that we know a priori what type of perturbation affects the control system, perhaps based on previous work or expert intuition. Once the perturbation is identified, the plant optimization problem (Eq. (5)) can be solved for different levels of perturbations. Then we can use numerical differentiation to obtain a measure of how optimal plant design changes as the control trajectories are perturbed. If the norm is calculated, the resulting control-plant coupling strength measure is:

\[
\left\| \frac{\partial x_p^*(u)}{\partial \alpha} \right\|_\infty,
\]

where \( \alpha \) is the amplitude of the perturbation applied to the control input. If the perturbation is a time delay or a phase shift, \( \alpha \) is a measure of the magnitude of the delay or phase shift. The two sets of equations described above (Eqs. (6)-(7) and Eqs. (8)-(9)) quantify coupling; the difference between these sets is that Eqs. (8) and (9) account for the time dependency of the control variables, whereas Eqs. (6) and (7) assume time-invariant control variables.

### 3. Design Coupling for Problem Formulation Decisions

Now we will discuss how the design coupling measure can be used to support CCD optimization problem formulation decisions. The design coupling strength measures derived in the previous section can be used to determine the nature of the coupling. Given a plant variable and a control variable, if both derivative norms (Eqs. (6)-(7) or Eqs. (8)-(9) are zero, there is no coupling between that pair of control and design variables. If one norm is zero and the other is not, then the coupling is unidirectional. Lastly, if both norms are greater than zero, then the coupling is bidirectional, and control and plant mutually affect each other, although not necessarily with the same strength.

One of the challenges of solving CCD problems arises when plant and control are optimized simultaneously to produce system-optimal solutions. This combined problem is typically more difficult to solve than when solved sequentially [12, 25]. In some scenarios, such as the case where computational resources are limited and the CCD problem is difficult to solve, the ability to solve the problem sequentially, and having a guarantee that the solution is still system-optimal, might be preferable to solving a CCD problem all-at-once or in an iterated sequential manner, even if a simultaneous solution achieves a better objective function value. If we are seeking to solve the CCD problem sequentially, when we formulate the optimization problem we should select variables such that there is minimal coupling between plant and control. When there is no coupling, the plant...
and control problems can be sequentially solved without compromising system optimality [16, 26]. In many CCD problems of practical importance, it may be impossible to structure coupling such that sequential solution is accurate.

If the optimization user is seeking to design a system that has a robust controller, they might want to select plant variables that have little effect on the optimal control inputs, so that the trajectories remain optimal with plant design variations. In practice, this means selecting variables such that the norm in Eq. (6) is as close to zero as possible. It should be noted that when we speak of robustness we are referring to the robustness of the controller to perturbations from the plant design variables. Changes in the plant, however, are not the only source of disturbances that a controller might experience. Lastly, if the goal is to select the most impactful design variable out of a pool of candidates, we can use the norms described in Sec. 2 to aid in the selection. If the optimal value of a plant variable remains constant with control changes, that is, the control-plant coupling is zero, it might make more sense to convert this variable to a fixed problem parameter, and instead choose a variable whose optimal value will change as we move in the plant-control design space. Additionally, if we seek to obtain an objective function value as low as possible, it might be necessary to compute additional information, such as \( \frac{\partial f}{\partial x_p} \), which indicates how the objective function changes with plant design changes, to make decisions about plant variable selection.

One concern that is not addressed in this work, but should be kept in mind when using design coupling strength to support problem formulation decisions, is scaling the variables. If, when comparing two variables, they typically move in ranges that are several orders of magnitude different, the resulting sensitivities may be difficult to interpret, and it might be appropriate to scale the variables (Ref. [18]). In the following example, the variables move in similar ranges, therefore scaling them was not deemed necessary.

4. EXAMPLE: SPACECRAFT CCD OPTIMIZATION

The following example is a CCD optimization problem of a spacecraft plant and control in Low Earth Orbit (LEO). This example shows the approach to estimate design coupling, applied to a practical engineering system. The spacecraft is pointing toward the Earth and is equipped with four reaction wheels to perform attitude maneuvers. It uses an LQR (Linear-Quadratic Regulator) controller to stabilize the spacecraft's attitude [27]. The roll, pitch, and yaw reaction wheels are aligned with the spacecraft body frame of reference, and the fourth wheel is at an angle with respect to the body frame. Figure 1 illustrates the spacecraft, reaction wheels, and the reference frames. The spacecraft dynamics are given by a system of nonlinear differential equations [28]. The degrees of freedom include the rotation of the spacecraft about its body axis \( \omega \), the quaternions \( q \), and the angular velocity of the reaction wheels \( \Omega \). The nonlinear equations are linearized around the equilibrium point \( \Omega = 0, q = 0, \omega = 0 \), resulting in a system of the form \( \dot{\xi} = A\xi(t) + Bu(t) \). The state vector is \( \xi = [\omega_1, \omega_2, \omega_3, \Omega_1, \Omega_2, \Omega_3, q_1, q_2, q_3]^T \), and the control input vector is the reaction wheel torques \( u(t) = [u_1, u_2, u_3, u_4]^T \). The initial state is \( \xi_0 = [0.0001, 0.0001, 0.0001, 0.0001, 0.0001, 0.0001, 0.0001, 0.0001, 0.0001, 0.0001, 0.0001]^T \). The linear system matrices are given in Eq. (11), where \( J \) denotes the spacecraft's moment of inertia, \( n \) is the rotational velocity of the spacecraft around the Earth, and \( J_e \) is the moment of inertia of each reaction wheel.

![Spacecraft in orbit with body frame, LVLH frame (Earth pointing), and inertial frame.](image)

The rotational velocity of the spacecraft is given by:

\[
\frac{1}{2\pi} \sqrt{\frac{\mu}{R_{sc}}},
\]

where \( \mu \) is the standard gravitational parameter and \( R_{sc} \) is the distance from the center of the Earth to the spacecraft.

The attitude control system must bring the spacecraft to its nominal orientation when it deviates. Suppose we want to optimize the plant and control design of the spacecraft such that it can return to its nominal state easily if disturbed. Here we will simultaneously optimize the control signals and a plant design variable: one of the dimensions of the satellite hub. Here we would like to select a single plant design variable to most efficiently improve system performance. To help identify the most appropriate plant design variable, we will estimate the design coupling between control design \( u_i \) and candidate plant design variables. The candidate plant design variables are the spacecraft width, \( w \), height, \( h \), and depth, \( d \). These plant variables affect the spacecraft dynamics through the moments of inertia \( I_i \), and are constrained by bounds due to practical reasons such as having enough space to fit subsystems and fitting inside the launch vehicle. In practice, there will additionally be an upper and lower bound on the control inputs and state values. Here we cannot explicitly add them since the controller is LQR, but by tuning the \( Q \) and \( R \) matrices we can ensure that the system response and control inputs stay within appropriate bounds. With that in mind, matrices are defined as \( Q = I_{10 \times 10} \) and \( R = 10^2I_{4 \times 4} \). Additionally, plant design variables must be chosen to ensure system controllability. The control co-design problem can be formulated...
The optimal control trajectory is given by:

\[ \text{dealing with the whole trajectories means that only the optimal gain needs to be found, we will be solving the optimal control problem. Even though the nominal controller is LQR, which simplifies the optimization problem, we must look at the control signals } \mathbf{u}(t) \text{ as a whole instead of only } \mathbf{K} \text{ if we want to capture a complete picture of how the control is affected by plant design changes. To look at the effect that changes in the control input have on the optimal plant design, we must look at } \mathbf{u}(t) \text{ instead of only } \mathbf{K} \text{ if we want to capture a complete picture of how the control is affected by plant design changes. To look at the effect that changes in the control input have on the optimal plant design, we first need to solve the optimization problem in Eq. (15):} \]

\[
\begin{align*}
\min_{\mathbf{x}_p, \mathbf{u}} & \quad \int_0^\infty \left( \mathbf{\dot{x}}^T \mathbf{Q} \mathbf{x} + \mathbf{u}^T \mathbf{R} \mathbf{u} \right) dt \\
\text{s.t.} & \quad V_{\min} - \text{w} h d \leq 0 \\
& \quad \mathbf{x}_{p,lb} \leq \mathbf{x}_p \leq \mathbf{x}_{p,ub} \\
& \quad (\mathbf{A}, \mathbf{B}) \text{ controllable} \\
& \quad \mathbf{\dot{x}} = \mathbf{A} \mathbf{x} + \mathbf{B} \mathbf{u} \\
\end{align*}
\]

where \( \mathbf{K} \) is the LQR gain. With this formula we can numerically estimate the derivative \( \partial \mathbf{u} / \partial \mathbf{x}_p \) via finite differences. Please note that additional design coupling complexity will be present in problems that do include general nonlinear inequality constraints in the optimal control problem, such as path constraints on actuator limits and plant failure modes. These additional aspects of plant-control design coupling are topics for future work.

Given that the control trajectories are time-varying, the derivatives calculated in Eq. (2) will be matrices of size \( n_p \times n_t \), where \( n_t \) is the number of time steps and \( n_p \) number of different plant values for which the problem in Eq. (13) was solved, instead of vectors, making it difficult to quantify and interpret coupling strength. The control signals \( \mathbf{u}(t) \) are parameterized here using a single finite quantity (K). This raises the question of why \( \partial \mathbf{u} / \partial \mathbf{x}_p \) and \( \partial \mathbf{x}_p / \partial \mathbf{u} \) need to be computed instead of \( \partial \mathbf{K} / \partial \mathbf{x}_p \) and \( \partial \mathbf{K} / \partial \mathbf{u} \). The reason for looking at the derivatives of and with respect to the time-varying control \( \mathbf{u}(t) \) is that the control inputs depend not only on the LQR gain but also on the system states, which in turn depend on the plant design. Therefore, we must look at \( \mathbf{u}(t) \) as a whole instead of only \( \mathbf{K} \) if we want to capture a complete picture of how the control is affected by plant design changes. To look at the effect that changes in the control input have on the optimal plant design, we first need to solve the optimization problem in Eq. (15):

\[
\begin{align*}
\min_{\mathbf{x}_p} & \quad \int_0^\infty \left( \mathbf{\dot{x}}^T \mathbf{Q} \mathbf{x} + \mathbf{u}^T \mathbf{R} \mathbf{u} \right) dt \\
\text{s.t.} & \quad V_{\min} - \text{w} h d \leq 0 \\
& \quad \mathbf{x}_{p,lb} \leq \mathbf{x}_p \leq \mathbf{x}_{p,ub} \\
& \quad (\mathbf{A}, \mathbf{B}) \text{ controllable} \\
& \quad \mathbf{\dot{x}} = \mathbf{A} \mathbf{x} + \mathbf{B} \mathbf{u}(t) \\
\end{align*}
\]

where \( \mathbf{K} \) is the LQR gain.
The optimization problem (15) can be solved for different control signals, and then the effect of control changes in optimal plant design can be estimated by calculating the derivative.

The process of solving the optimization problems in Eqs. (13) and (15) will now be detailed for the case where \( w \) is the plant design variable. This process is repeated for the other plant variables, but is not explained again for brevity. To see how changing the width of the spacecraft affects the optimal control variables, but is not explained again for brevity. To see how changing the width of the spacecraft affects the optimal control trajectories, the optimal control problem (Eq. (13)) is solved for different \( w \) values, with the rest of the plant variables being fixed to their nominal values, and the derivative has been calculated to estimate coupling. Figure 2 shows the optimal control inputs for different plant parameter \( w \) values on the first row. On the second row we have the derivatives of the optimal control inputs with respect to \( w \). Since the control inputs are trajectories that change with time, the partial derivatives also change with time, producing a surface.

To better visualize the effect of plant changes in optimal control input, the surfaces in Fig. 2 are converted to curves (Fig. 4) using the process described in Sec. 2 and illustrated in Fig. 3. We go from optimal control inputs that depend on plant design and time to a single quantity that measures the effect of width changes in optimal control trajectory. There is a jump when \( w = 0.9 \); this is because some of the elements of the \( A \) matrix become zero, and the system becomes uncontrollable. Notably, changes in the spacecraft width affect the control inputs differently, having a larger effect on the first and fourth control inputs.

To study the effect of control changes on optimal plant design we will look at three types of control perturbations. While it is possible that several disturbances are present at one time, here we assume that only one disturbance is present in each case to simplify the analysis. The nominal controller is LQR, with the control inputs having the form \( u(t) = -K\xi(t) \), because we are perturbing the control inputs, the true inputs might be a partial consequence of faults and hence will not follow that formula. During a spacecraft’s lifetime, sensors and actuators may malfunction or stop working entirely. If the attitude determination sensors malfunction, they will provide a non-accurate representation of the system state to the controller, which may result in the calculation of a gain matrix that is not optimal. One simple way to simulate miscalculations of the gain matrix is to multiply some or all of its elements by a scalar. The effect of perturbing the gain matrix is shown in Fig. 5, where the optimal plant parameter value \( w^* \) has been calculated for slightly different control gain matrices. Specifically, the gain matrix has been kept at its nominal value, but one of its elements has been multiplied by a scalar, \( \alpha \). As can be seen, when there are changes in the gain matrix, the optimal plant design will change. The vector norm has been calculated to obtain a measure of the control-plant coupling strength.

The other two control perturbations that are considered here are inspired by fault-tolerant control [29, 30]. First, we will look at the effect of a control bias, then at a reaction wheel fault where the wheel has reduced control torque. The control input with a bias is modeled as:

\[
u_{bias} = \begin{cases} u + 0.021\alpha, & 10 < t \leq 50 \text{ s}, \\ u, & \text{otherwise}, \end{cases} \tag{16}
\]

where \( \alpha \) quantifies how large the bias is. The bias has been formulated so that, at most, it is 15% of the maximum allowed torque. For the case where the system has reduced control torque, the control input is given by:

\[
u_{\text{reduced}} = \begin{cases} u\alpha, & 10 < t \leq 30 \text{ s}, \\ u, & \text{otherwise}, \end{cases} \tag{17}
\]

where \( \alpha \) quantifies the reduction in control torque. The control fault functions in Eqs. (16) and (17) are non-smooth functions. Since all these studies are being performed with a gradient-based optimization algorithm, we have approximated both disturbances as sigmoid functions, which are smooth, to help solver convergence.

The previous process of calculating the optimal spacecraft width for different gain matrices is then repeated for the other two perturbations. Control torque bias, loss, and errors in the gain matrix calculation are not the only cases of interest. For instance, other possible changes in the control could include time delays, but studying their effect on the optimal plant design is left for future work.

This process of calculating the coupling between the width and control torques is repeated for the other two plant variables, depth and height. The coupling strength between plant and control is shown in Tables 1 and 3. Looking at the results in Table 3, we can see that the optimal depth and height do not change with changes in the control input. However, the optimal width is sensitive to changes in the control. In terms of how plant changes affect optimal control, we see that changes in any of the plant variables affect the optimal control trajectories. Therefore, we can say that there is bidirectional coupling between control design and width, but the coupling between depth and height and control is uni-directional, with the plant affecting the optimal control design.

This information can now help inform problem formulation decisions. In the case that we have a nominal spacecraft design that we want to improve but have limited computational resources and would like to optimize only one plant parameter, we would like to choose the design variable that will have the most impact. If we look at Table 1 most of the coupling strengths have similar orders of magnitude. When we look at Table 3, however, the coupling strengths vary significantly. As mentioned above,
FIGURE 2: First row: Optimal control trajectories for different spacecraft width values. Maximum allowed torque input represented in dashed line. Second row: optimal torque derivative with respect to width.

FIGURE 3: Plant-control coupling calculation process.

FIGURE 4: Control input integral (left) and derivative of the input integral with respect to spacecraft width (right)

FIGURE 5: Optimal spacecraft width (left) and derivative of the spacecraft width with respect to control gain change $\alpha$ (right)

width design is more sensitive than the other two plant variables to changes in control input. Since the optimal depth and height do not appear to change with changes in control, setting these as problem parameters and retaining width as the optimization variable may make sense. This prediction based on coupling strength evaluation is confirmed by testing problem solution for each of

TABLE 3: Control-plant coupling strength for different control perturbations

<table>
<thead>
<tr>
<th>$u + 0.021\alpha$</th>
<th>$u\alpha$</th>
<th>$K_{13} = K_{13}\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\partial u^*_w(u)}{\partial u}$</td>
<td>$w^*$</td>
<td>$d^*$</td>
</tr>
<tr>
<td>$u_1$</td>
<td>3.21\times10^{-3}</td>
<td>9.81\times10^{-14}</td>
</tr>
<tr>
<td>$u_2$</td>
<td>3.76\times10^{-3}</td>
<td>6.18\times10^{-15}</td>
</tr>
<tr>
<td>$u_3$</td>
<td>4.57\times10^{-3}</td>
<td>3.52\times10^{-14}</td>
</tr>
<tr>
<td>$u_4$</td>
<td>5.30\times10^{-3}</td>
<td>5.59\times10^{-14}</td>
</tr>
</tbody>
</table>

$u^*(t) \rightarrow \int_0^\infty u^2(t) dt \rightarrow \frac{\partial \int_0^\infty u^2(t) dt}{\partial x_p} \rightarrow \frac{\partial \int_0^\infty u^2(t) dt}{\partial x_p}$
TABLE 4: CCD optimization objective function values with only one plant design variable, problems solved sequentially

<table>
<thead>
<tr>
<th>Design variable</th>
<th>$w$</th>
<th>$d$</th>
<th>$h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(\cdot)^*$</td>
<td>1.30·10^5</td>
<td>1.83·10^5</td>
<td>8.64·10^5</td>
</tr>
</tbody>
</table>

the three candidate design variables (see Table 2). The lowest objective function value is achieved when the CCD problem is solved with $w$ as the design variable, and $d$ and $h$ are set as fixed problem parameters.

If we are instead seeking to solve the problem sequentially, we need to select plant and control design variables with minimal uncoupling to improve accuracy of the sequential solution strategy. In this case, there are no plant variables that are completely uncoupled. As stated in Ref. [26], if the coupling is unidirectional and the problem is solved sequentially, the design will be optimal for the plant, but it is not for the controller. Table 4 summarizes the suboptimal results obtained through sequential solution. If the plant design variable is the width and the problem is solved sequentially, the objective function value is an order of magnitude higher than when solved in a combined manner; we expected such a result since the width-control coupling is bidirectional. When the plant variable is the depth, the objective is also higher when solving sequentially, but less so than in the case of width. This could be because the depth-control coupling, at least in this formulation, is unidirectional. Lastly, even though the height-control designs have unidirectional coupling, the same minimum is achieved when the problem is solved sequentially.

5. CONCLUSION

In this work, we have presented a method to estimate coupling strength in CCD optimization problems, specifically accounting for bidirectional coupling and providing a strategy for estimating coupling when the control variables are time-varying trajectories. This method is demonstrated with an example, where the coupling strength between plant and control design has been estimated for a spacecraft design optimization problem. The coupling information can be useful for supporting optimization problem formulation decisions, and some insights have been provided. In large-dimension optimization problems, we can use design coupling strength as a strategy to help reduce the number of design variables needed to obtain a good optimization result by retaining only the most impactful variables. The coupling strength may also be used to assess whether the CCD problem could be solved sequentially without significant error. That said, guidelines for this decision will be problem dependent, will require significant additional investigation, and is a topic for future work. Additionally, if robustness is a concern, the designer could use the coupling information to help identify designs such that the optimal control trajectory design is not very sensitive to changes in the plant. That said, consideration of system robustness in the context of CCD should be made with a holistic perspective of uncertainty (beyond robust control and robust design treated separately).

One drawback of this method is that it requires solving the optimization problem to calculate the derivatives, and it is of interest to investigate other methods to measure bidirectional coupling strength without needing to solve the optimization problem first. Additional future work includes extending this coupling measure to deal with all possible kinds of control perturbations, including time delays.

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REFERENCES


