Online Inner Approximation of Reachable Sets of Nonlinear Systems with Diminished Control Authority

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Abstract

This work presents a method of efficiently computing inner approximations of forward reachable sets for nonlinear control systems with diminished control authority, given an a priori computed reachable set for the nominal system. The method functions by shrinking a precomputed convex reachable set based on a priori knowledge of the system's trajectory deviation growth dynamics. The trajectory deviation growth dynamics determine an upper bound on the minimal deviation between two trajectories emanating from the same point that are generated by control inputs from the nominal and diminished set of control inputs, respectively. These growth dynamics are a function of a given Hausdorff distance bound between the nominal convex space of admissible controls and the possibly unknown impaired space of admissible controls. Because of its relative computational efficiency compared to direct computation of the off-nominal reachable set, this procedure can be applied to onboard fault-tolerant path planning and failure recovery. We consider the implementation of the approximation procedure by way of numerical integration and a root finding scheme, and we present two illustrative examples, namely an application to a control system with quadratic nonlinearities and aircraft wing rock dynamics.

1 Introduction.

Reachability analysis forms a fundamental part of dynamic system analysis and control theory, providing a means to assess the set of states that a system can reach under admissible control inputs at a certain point in time from a given set of initial states. Inner approximations of reachable sets are often used to attain a guaranteed estimate of the system's capabilities, while outer approximations can be used to verify that the system will not reach an unsafe state. Such outer approximations find widespread applications in fault-tolerance analysis and formal verification [5], as well as safe trajectory planning [27]. Methods for computing outer approximations of the reachable set include polynomial overapproximation [1] and viscosity solutions to the Hamilton–Jacobi–Bellman (HJB) equations [4].

Inner approximations of reachable sets have received comparatively less attention than outer approximations [10], but have recently seen use in path-planning problems with collision avoidance [25], as well as viability kernel computation [11], which can in turn be used for guaranteed trajectory planning [13]. Another application is flight envelope estimation for aircraft; the penalty of overconfidence in flight envelope estimation is often severe, and over-preparedness as the cost of underconfidence is much preferred in such a context [26]. Methods for determining inner approximations of reachable sets have been based on various principles, including relying on polynomial inner approximation of the nonlinear system dynamics using interval calculus [9], ellipsoid calculus [8], and viscosity solutions to HJB equations [29]. One major drawback of these methods is that they are computationally intensive and are often only suitable for systems of low-dimension, making them ill-suited for online use.

Motivated by the desire to leverage available a priori information on the nominal system dynamics and trajectory deviation growth dynamics, this paper focuses on finding an inner approximation of the reachable set using this information rather than starting from scratch. Here, we consider the reachable set of the nominal system, or an inner approximation thereof, to be known prior to the system's operation. As noted before, obtaining this reachable set is often a computationally intensive task, yet it is crucial that a reachable set be obtained in safety-critical applications; for this reason, the reachable set is often obtained during the design phase of a system [16]. We then consider a change in dynamics of the system, for example due to partial system failure, which turns the nominal system into the *off-nominal* system. In particular, we consider the case in which the system experiences *diminished control authority*, i.e., its set of admissible control inputs has shrunk with respect to that of the nominal system. We consider that an upper bound on the minimal rate of change of the trajectory deviation between the off-nominal system's trajectories with respect to those of the nominal system's is known, with both trajectories emanating from the same point. These growth dynamics provide an upper bound on the minimal rate of change between two trajectories emanating from the same point, with one trajectory being generated by the nominal set of control inputs, and the other by the off-nominal set of control inputs. An upper bound on these growth dynamics can be obtained analytically during the de-

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sign phase, and allows us to obtain an inner approximation to the off-nominal system's reachable set at low cost, in an online manner. Our method applies to affine-in-control systems under some technical assumptions.

While methods have been proposed to compute reachable sets under system impairment, due to their computational complexity, these have either used reduced order models, or have been limited to offline applications [20]. Given a sufficiently tight bound on the trajectory deviation growth dynamics, our approach can be applied online to higher dimensional systems with no additional computational cost for the growth in system dimension. To the best of our knowledge, an approach similar to ours has not been considered in the literature.

This paper is organized as follows. In Sec. 2, we formally present the problem of inner approximation of the reachable set and introduce the notion of diminished control authority, as well as sufficient conditions for reachable set convexity. Sec. 3 presents an integral inequality that provides a general bound on the trajectory deviation growth between trajectories of the nominal and off-nominal system emanating from the same point. Sec. 4 contains the main results, providing the means of inner approximating the off-nominal reachable set based on the nominal reachable set and known trajectory deviation growth conditions. We illustrate the theory in Sec. 5, where we apply it to a control system with quadratic nonlinearities and an aircraft wing rock model.

Notation. In the following, we denote by $\langle\cdot,\cdot\rangle$ the inner product, i.e., $\langle a, b\rangle = a^\mathsf{T} b$ for $a, b \in \mathbb{R}^n$. By $\|\cdot\|$ we denote the Euclidean norm. We denote a ball centered around the origin with radius $r > 0$ as $\mathcal{B}_r$. By $\mathcal{B}(x, r)$ we denote $x + \mathcal{B}_r$. We define $\mathbb{R}_+ := [0, \infty)$. We denote the Hausdorff distance between two sets $A, B \subseteq S$ as

$$d_\mathrm{H}(A, B) := \max\{\sup_{a\in A}\inf_{b\in B} d(a, b), \sup_{b\in B}\inf_{a\in A} d(a, b)\},$$

where S is a given metric space with metric d. Given a point $x \in S$ and a set $A \subseteq S$, we denote $d(x, A) := \inf_{y\in A} d(x, y)$. By $A + B$ we denote Minkowski sum $\{a + b : a \in A, b \in B\}$. We denote by ∂A the boundary of A. For a function $g : A \to B$, we denote by g^{-1} the inverse of this function if an inverse exists, and by $\mathrm{dom}(g)$ the domain of the function (in this case A). We denote a multifunction by $G : A \rightrightarrows B$, where G maps elements of A to subsets of B.

2 Problem Formulation.

Consider a dynamical system of the form

$$\dot{x}(t) = f(t, x(t), u(t)), \quad x(0) = x_0, \tag{2.1}$$

where $t_0 \in \mathbb{R}_+$ is a given initial time, $t \in [t_0, \infty)$, $x \in \mathbb{R}^n$ is the state, and $u \in \mathcal{U} \subseteq \mathbb{R}^m$ is the control input, where $\mathcal{U}$ is some admissible set of control inputs. We refer to dynamics (2.1) as the *nominal* dynamics.

We consider a modified set of admissible control inputs $\bar{\mathcal{U}} \subseteq \mathcal{U}$, reflecting a reduction in control authority. The modified dynamics then read

$$\dot{\bar{x}}(t) = f(t, \bar{x}(t), \bar{u}(t)), \quad \bar{x}(t_0) = x_0, \tag{2.2}$$

with $\bar{u} \in \bar{\mathcal{U}}$.

We refer to dynamics (2.2) as the *off-nominal* dynamics. Our goal is to efficiently find an inner approximation of the forward reachable set with off-nominal dynamics (2.2), under some assumptions on the dynamics and the admissible sets of controls $\bar{\mathcal{U}}, \mathcal{U}$, *given* the forward reachable set with nominal dynamics (2.1) from the same set of initial points.

We first lay out a number of definitions.

Definition 2.1. (Trajectories) *A function $\phi : \mathbb{R} \to \mathcal{U}$ is known as an* admissible input signal. *The set of admissible control signals comprises all possible admissible input signals, i.e., $\mathbb{U} := \{\phi : \mathbb{R} \to \mathcal{U} \mid$ a solution to (2.1) for ϕ exists and is unique$\}$.*

A trajectory $\varphi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^n$ *is defined such that $x(t) = \varphi(t|t_0, x_0, \phi)$ satisfies (2.1) given initial time $t_0 \in \mathbb{R}_+$, initial state $x(t_0) = x_0 \in \mathbb{R}^n$, and input signal $u(t) = \phi(t) \in \mathbb{U}$, i.e., $\varphi(t|t_0, x_0, \phi) := x_0 + \int_{t_0}^{t} f(\tau, x(\tau), \phi(\tau))\, \mathrm{d}\tau$.*

By definition, any trajectory $\varphi(t|t_0, x_0, \phi)$ satisfies $\dot{\varphi}(t|t_0, x_0, \phi) = f(t, \varphi(t|t_0, x_0, \phi), \phi(t)) \in F(t, \varphi(t|t_0, x_0, \phi))$, where $F(t, x) := f(t, x, \mathcal{U})$, for all $t \in \mathbb{R}_+$.

All of the above definitions are similarly extended for (2.2), with all symbols having an overbar.

Definition 2.2. (Forward reachable set) *Let $F(t, x) := f(t, x, \mathcal{U})$. Given a set of initial states $\mathcal{X}_0 \subseteq \mathbb{R}^n$ and an initial time $t_0 \in \mathbb{R}_+$, the* forward reachable set *(FRS) at time $\tau \in [t_0, \infty)$ is*

$$\mathbb{X}^\rightarrow_\tau = \mathbb{X}^\rightarrow_\tau(F, \mathcal{X}_0) := \{\varphi(\tau|t_0, x_0, \phi) : x_0 \in \mathcal{X}_0, \phi \in \mathbb{U}\}.$$

Problem 1. (Off-nominal FRS inner approximation) *Given dynamics $\dot{x}(t) = f(t, x(t), u(t))$, where $f : [0, \infty) \times \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n$, a set of admissible control inputs $\mathcal{U} \subset \mathbb{R}^m$, (an inner approximation of the) forward reachable set $\mathbb{X}^\rightarrow_\tau$ at time τ, and the corresponding initial set of states $\mathcal{X}_0$ and an initial time $t_0 \in \mathbb{R}_+$, as well as a Hausdorff distance bound $\epsilon > 0$ such that $d_\mathrm{H}(\mathcal{U}, \bar{\mathcal{U}}) \leq \epsilon$, find an inner approximation of the reachable set $\bar{\mathbb{X}}^\rightarrow_\tau$ for dynamics (2.2).*

We emphasize that Problem 1 only requires knowledge of an upper bound on the Hausdorff distance between the nominal and off-nominal set of admissible control inputs, i.e., $d_\mathrm{H}(\mathcal{U}, \bar{\mathcal{U}}) \leq \epsilon$; the set of off-nominal admissible control inputs $\bar{\mathcal{U}}$ itself need not be known.

2.1 Conditions for Reachable Set Convexity. We proceed by recalling a number of sufficient conditions for convexity of reachable sets. Before we do so, we present a definition of a stronger convexity condition on sets.

DEFINITION 2.3. (R-CONVEXITY [14, P. 124], [15, P. 191]) *Given $R \geq 0$, a nonempty compact set $A \subset \mathbb{R}^k$ is R-*convex *if A is the intersection of closed balls of radius R. The number of balls that are intersected need not be finite or countable.*

R-convex sets are known to be *strictly convex*, i.e., their boundary does not contain any line segments between any two members [14, p. 124]. The family of R-convex encompasses many sets frequently used in control theory, including balls and ellipsoids. Even though other convex sets such as hyperrectangles are not R-convex for any $R \in \mathbb{R}_+$, the theory we are presenting appears to extend to some of these cases, as shown in Sec. 5.2.

The following definition will be used in the main theorem of this section, which provides an explicit interval of existence of a convex reachable set.

DEFINITION 2.4. (PLIŚ METRIC [22, EQ. 3]) *Given two strictly convex compact sets $A, B \subseteq \mathbb{R}^k$, the* Pliś metric *is defined as $d_{\mathrm{P}}(A, B) := \max_{p \in \partial \mathcal{B}_1} \|v(A, p) - v(B, p)\|$, where $v(X, p)$ satisfies $\langle p, v(X, p)\rangle = \max_{x \in X} \langle p, x\rangle$ for any strictly convex compact set X.*

THEOREM 2.1. (FRS CONVEXITY [15, THM. 2]) *Let F be a continuous multifunction $\mathbb{R} \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, with $n \geq 2$. Let there exist constants $R, M, L \geq 0$, and assume that for all $t \in \mathbb{R}$, $x, y \in \mathbb{R}^n$, and $\lambda \in [0, 1]$:*

(1) $F(t, x)$ is R-convex;
(2) $d_{\mathrm{P}}(F(t, x), F(t, y)) \leq M\|x - y\|$;
(3) $d_{\mathrm{P}}(F(t, \lambda x + (1 - \lambda)y), \lambda F(t, x) + (1 - \lambda)F(t, y)) \leq L\lambda(1 - \lambda)\|x - y\|^2$.

Let F be measurable with respect to $t \in \mathbb{R}$ for each $x \in \mathbb{R}^n$, and let it be bounded by an integrable function on each compact set in $\mathbb{R} \times \mathbb{R}^n$. Let $\mathcal{X}_0 \subset \mathbb{R}^n$ be R_0-convex. Let $\gamma(\cdot)$ be a solution to the differential Riccati equation $\dot{\gamma} = L + 3M\gamma + 2R\gamma^2$, with $\gamma(t_0) = R_0$, and define $\Delta t := \int_{R_0}^{\infty} (R + 3M\gamma + 2L\gamma^2)^{-1} \mathrm{d}\gamma$.

Then, $\Delta t > 0$ and $\mathbb{X}_t^{\rightarrow}(F, \mathcal{X}_0)$ is $\gamma(t)$-convex for all $t \in [t_0, t_0 + \Delta t]$.

3 Trajectory Deviation Growth Bounds.

To be able to relate the trajectories of the nominal system to those of the off-nominal system, it is necessary that we have knowledge of the *growth of the trajectory deviation*. Specifically, we must know how far apart two trajectories, emanating from the same point, may at least grow as a function of time, given that one trajectory is generated using nominal control inputs, and the other is generated using off-nominal control inputs. In this section, we present an integral inequality that uses only a bound on the trajectory deviation dynamics as a function of time.

We assume $\mathcal{U} \subset \mathbb{R}^m$ to be a compact set, i.e., there exists a $\delta \geq 0$ such that $\max_{u \in \mathcal{U}} \|u\| = \delta$. To obtain the desired bounds on the trajectory deviation, we make the following assumptions on some general function $h : [t_0, \infty) \times \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n$, which will later be taken as the difference between the dynamics of the nominal and off-nominal system.

We now pose a nonlinear bound on the magnitude of h:

ASSUMPTION A1. *For a function $h : [t_0, \infty) \times \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n$, assume $\|h(t, x, u)\| \leq a(t)w(\|x\|, \|u\|) + b(t)$ for all $t_0 \leq t < \infty$, where a, b are continuous and positive and w is continuous, positive and nondecreasing in $\|x\|$ and $\|u\|$.*

We can now prove the following integral inequality, which is an extension of the Bihari inequality [21, p. 113, Thm. 2.3.4] in that we consider systems with control inputs.

THEOREM 3.1. (EXTENDED BIHARI INEQUALITY) *Let $x(t)$ be a solution to the equation*

$$\dot{x} = h(t, x, u), \quad 0 \leq t_0 \leq t < \infty,$$

where $h(t, x, u) : [t_0, \infty) \times \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n$ is continuous for $t_0 \leq t < \infty$, and $\mathcal{U} \subset \mathbb{R}^m$ is compact and satisfies $\max_{u \in \mathcal{U}} \|u\| = \delta$. Let A1 hold. Then,

(3.3)
$$\|x(t)\| \leq G^{-1}\left[G\left(\|x(t_0)\| + \int_{t_0}^{t} b(\tau)\mathrm{d}\tau\right) + \int_{t_0}^{t} a(\tau)\mathrm{d}\tau\right],$$

where the upper bound is strictly increasing in t, and

$$G(r) := \int_{r_0}^{r} \frac{\mathrm{d}s}{w(s, \delta)}, \quad r > 0, r_0 > 0,$$

for arbitrary $r_0 > 0$ and for all $t \geq t_0$ for which it holds that

$$G\left(\|x(t_0)\| + \int_{t_0}^{t} b(\tau)\mathrm{d}\tau\right) + \int_{t_0}^{t} a(\tau)\mathrm{d}\tau \in \mathrm{dom}(G^{-1}).$$

Proof. From (2.1), we have that

$$x(t) = x(t_0) + \int_{t_0}^{t} f(\tau, x(\tau), u(\tau))\, \mathrm{d}\tau,$$

from which we find, after applying A1,

$$\|x(t)\| \leq \|x(t_0)\| + \int_{t_0}^{t} b(\tau)\mathrm{d}\tau + \int_{t_0}^{t} a(\tau)w(\|x(\tau)\|, \|u(\tau)\|)\mathrm{d}\tau.$$

We define functions y, g as

$$y(t) := \int_{t_0}^{t} a(\tau)w(\|x(\tau)\|, \|u(\tau)\|)\mathrm{d}\tau, \quad t \geq t_0,$$

$$g(t) := \|x(t_0)\| + \int_{t_0}^{t} b(\tau)\mathrm{d}\tau, \quad t \geq t_0,$$

which gives $\|x(t)\| \le g(t) + y(t)$ for $t \ge t_0$. Differentiating these functions, we find

$$y'(t) = a(t)w(\|x(t)\|, \|u(t)\|) \le a(t)w(g(t)+y(t), \delta), \quad t \ge t_0,$$
$$g'(t) = b(t), \quad t \ge t_0,$$

which gives

$$\begin{aligned} \frac{g'(t)+y'(t)}{w(g(t)+y(t),\delta)} &\le a(t) + \frac{g'(t)}{w(g(t)+y(t),\delta)} \\ &\le a(t) + \frac{g'(t)}{w(g(t),\delta)}, \quad t \ge t_0, \end{aligned} \tag{3.4}$$

where the inequalities follow from the positivity and monotonicity of $y'(t)$ and $w(\cdot, \delta)$. Integrating (3.4) yields

$$\int_{g(t_0)+y(t_0)}^{g(t)+y(t)} \frac{\mathrm{d}s}{w(s,\delta)} \le \int_{t_0}^{t} a(s)\mathrm{d}s + \int_{g(t_0)}^{g(t)} \frac{\mathrm{d}s}{w(s,\delta)}, \tag{3.5}$$

for $t \ge t_0$. Noting that $y(t_0) = 0$ and $g(t_0) = \|x(t_0)\|$, from (3.5) we obtain

$$\begin{aligned} &G(g(t)+y(t)) - G(\|x(t_0)\|) \le \\ &\int_{t_0}^{t} a(\tau)\mathrm{d}\tau + G(g(t)) - G(\|x(t_0)\|), \end{aligned}$$

which yields

$$G(g(t)+y(t)) \le G\left(\|x(t_0)\| + \int_{t_0}^{t} b(\tau)\mathrm{d}\tau\right) + \int_{t_0}^{t} a(\tau)\mathrm{d}\tau. \tag{3.6}$$

From (3.6) and $\|x(t)\| \le g(t) + y(t)$, applying the inverse of G, we obtain the inequality as in (3.3). Since G is strictly increasing because of the nonnegativity of a, b, w, G^{-1} is also strictly increasing by [24, p. 137, Thm. 18.4]. Therefore, the right-hand side of (3.3) is strictly increasing in t. □

Note that the results in Thm. 3.1 do not depend on the dimension of the actual system since only scalar values that represent vector norms are considered in the trajectory deviation growth bound. We are now in a position to make a claim about the difference between trajectories of the nominal and off-nominal system, given the assumptions on the growth conditions:

Corollary 3.1. *For any $x_0 \in \mathcal{X}_0 \subset \mathbb{R}^n$, where $\mathcal{X}_0$ is compact, and any initial time $t_i \in [t_0, \infty)$ and final time $t_f \in [t_i, \infty)$, consider a trajectory $x(t)$ satisfying $x(t_i) = x_0$ and $\dot{x}(t) = f(t, x(t), u(t))$, with $u(t) \in \mathbb{U}$. Consider a trajectory $\bar{x}(t)$ with $\bar{x}(t_i) = x_0$ and $\dot{\bar{x}}(t) = f(t, \bar{x}(t), \bar{u}(t))$, such that $\bar{u}(t) \in \bar{\mathbb{U}}$ satisfies $\sup_{t\in[t_i,t_f]} \|u(t) - \bar{u}(t)\| \le \epsilon$. Let $\tilde{f}(t) := f(t, x(t), u(t)) - f(t, \bar{x}(t), \bar{u}(t))$, $\tilde{x}(t) := x(t) - \bar{x}(t)$, and $\tilde{u}(t) := u(t) - \bar{u}(t)$. Let the following bound hold for $t \in [t_i, t_f]$, and for* any *$x(t)$ and $\bar{x}(t)$ satisfying the previous hypotheses: $\|\tilde{f}(t)\| \le \tilde{a}(t)\tilde{w}(\|\tilde{x}(t)\|, \|\tilde{u}(t)\|) + \tilde{b}(t)$ with $\tilde{a}, \tilde{b}, \tilde{w}$ satisfying the assumptions given in A1. Then, $\tilde{x}(t)$ satisfies*

$$\|\tilde{x}(t)\| \le G^{-1}\left[G\left(\int_{t_0}^{t} \tilde{b}(\tau)\mathrm{d}\tau\right) + \int_{t_0}^{t} \tilde{a}(\tau)\mathrm{d}\tau\right] =: \eta(t, \epsilon) \tag{3.7}$$

for all $t \in [t_i, t_f]$.

Proof. Given the premise, this claim follows directly from Thm. 3.1. □

4 Inner Approximation of the Reachable Set.

We now state the main result of this paper, which draws upon the trajectory deviation bound of Cor. 3.1 and the convexity guarantee of Thm. 2.1. In particular, we obtain a Hausdorff distance bound between the nominal and off-nominal reachable sets that holds during a time interval in which both reachable sets are guaranteed to be convex. This bound allows us to obtain an inner approximation of the off-nominal reachable set by shrinking the nominal reachable set by this Hausdorff distance bound.

Theorem 4.1. (Off-nom. FRS inner approximation) *Let $R, R', R_0 \ge 0$, and let $f : [0, \infty) \times \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n$, where $\mathcal{U} \subset \mathbb{R}^m$ is an R-convex set of admissible controls. Let $\mathcal{U}_\epsilon \subseteq \mathcal{U}$ be an R-convex set of off-nominal admissible controls such that $d_{\mathrm{H}}(\mathcal{U}, \mathcal{U}_\epsilon) \le \epsilon$. Let $F_\epsilon(t, x) = f(t, x, \mathcal{U}_\epsilon)$ and $F(t, x) = f(t, x, \mathcal{U})$ be R'-convex for all $(t, x) \in [t_0, \infty) \times \mathbb{R}^n$. Let an R_0-convex set $\mathcal{X}_0 \subset \mathbb{R}^n$ of initial states, and initial time $t_0 \in \mathbb{R}_+$ be given. Let the hypotheses of Cor. 3.1 be satisfied with $\bar{\mathcal{U}} = \mathcal{U}_\epsilon$, $t_i = t_0$, and t_f to be specified in claim (ii). Let $\eta(t, \epsilon)$ be obtained as in Cor. 3.1. Then:*

(i) there exists a $\Delta t > 0$ such that the reachable sets $\mathbb{X}_t^{\rightarrow}(F, \mathcal{X}_0)$ and $\mathbb{X}_t^{\rightarrow}(F_\epsilon, \mathcal{X}_0)$ are convex for $t \in [t_0, t_0 + \Delta t]$;

(ii) let $0 < T \le \Delta t$, and let $t_f \ge t_0 + T$. For each $x_0 \in \mathcal{X}_0$ there exists a trajectory $x(t)$ emanating from $x(t_0) = x_0$ with $\dot{x}(t) \in F(t, x(t))$ and a trajectory $x_\epsilon(t)$ satisfying $x_\epsilon(t_0) = x_0$ and $\dot{x}_\epsilon(t) \in F_\epsilon(t, x_\epsilon(t))$ such that $\|x(t) - x_\epsilon(t)\| \le \eta(t, \epsilon)$ for all $t \in [t_0, t_f]$;

(iii) for all $t \in [t_0, t_0 + \Delta t]$, $\mathbb{X}_t^{\rightarrow}(F_\epsilon, \mathcal{X}_0) \subseteq \mathbb{X}_t^{\rightarrow}(F, \mathcal{X}_0)$;

(iv) Let $\eta(\epsilon) = \eta(t_0 + T, \epsilon)$. For all $t \in [t_0, t_0 + T]$, $d_{\mathrm{H}}[\mathbb{X}_t^{\rightarrow}(F_\epsilon, \mathcal{X}_0), \mathbb{X}_t^{\rightarrow}(F, \mathcal{X}_0)] \le \eta(\epsilon)$;

(v) for all $t \in [t_0, t_0 + T]$,

$$\mathbb{X}_t^{\rightarrow}(F, \mathcal{X}_0) \setminus \bigcup_{x \in \partial \mathbb{X}_t^{\rightarrow}(F, \mathcal{X}_0)} \mathcal{B}(x, \eta(\epsilon)) \subseteq \mathbb{X}_t^{\rightarrow}(F_\epsilon, \mathcal{X}_0).$$

Proof. (i) By our hypotheses, both $F(t, x)$ and $F_\epsilon(t, x)$ are R'-convex for all $(t, x) \in [t_0, \infty) \times \mathbb{R}^n$. In addition, F_ϵ

shares constants M and L of Thm. 2.1 with F, since we have $F_\epsilon(t,x) \subseteq F(t,x)$ for all $(t,x) \in [t_0,\infty) \times \mathbb{R}^n$. Therefore, Thm. 2.1 is satisfied for both F and F_ϵ, which then share the same $\Delta t > 0$ such that for all $t \in [t_0, t_0+\Delta t]$ both $\mathbb{X}_t^{\rightarrow}(F_\epsilon, \mathscr{X}_0)$ and $\mathbb{X}_t^{\rightarrow}(F, \mathscr{X}_0)$ are convex.

(ii) This fact follows directly from Cor. 3.1.

(iii) This fact follows trivially by considering that any trajectory that can be generated using control inputs from a set of admissible controls ($\mathscr{U}_\epsilon$) can also be generated using control inputs from a superset ($\mathscr{U}$).

(iv) From (ii), the maximal distance between two points in $\mathbb{X}_t^{\rightarrow}(F, \mathscr{X}_0)$ and $\mathbb{X}_t^{\rightarrow}(F_\epsilon, \mathscr{X}_0)$ is given to be $\eta(t,\epsilon)$. In Thm. 3.1 it shown that $\eta(t,\epsilon)$ is increasing in t, which proves the claim.

(v) We shall now denote $\eta = \eta(\epsilon)$, and $A = \mathbb{X}_t^{\rightarrow}(F_\epsilon, \mathscr{X}_0)$ and $B = \mathbb{X}_t^{\rightarrow}(F, \mathscr{X}_0)$ for any $t \in [t_0, t_0+T]$; recall that A and B are convex. We wish to show that $B \setminus \left(\bigcup_{x\in\partial B} \mathcal{B}(x,\eta)\right) \subseteq A$.

We invoke a characterization of the Hausdorff distance [19, pp. 280–281]:

$$d_{\mathrm{H}}(A,B) = \inf\{\rho \geq 0 \,:\, A \subseteq B_{+\rho}, B \subseteq A_{+\rho}\}, \tag{4.8}$$

where $X_{+\rho}$ denotes the *ρ-fattening* of X, i.e., $X_{+\rho} := \bigcup_{x\in X}\{y \in \mathbb{R}^n \,:\, \|x-y\| \leq \rho\}$. Therefore, $d_{\mathrm{H}}(A,B) \leq \eta$ implies $B \subseteq A_{+\eta}$.

By [28, p. 116, Thm. 20], for two non-empty convex compact sets A, B, their Hausdorff distance equals the Hausdorff distance between their boundaries, which implies $d_{\mathrm{H}}(A,B) = d_{\mathrm{H}}(\partial A, \partial B) \leq \eta$ in our case, by noting that A, B are $\gamma(t)$-convex sets by Thm. 2.1, and are therefore compact by Def. 2.3. Applying the previous characterization of the Hausdorff distance, we find that $\partial A \subseteq (\partial B)_{+\eta}$ and $\partial B \subseteq (\partial A)_{+\eta}$.

We now prove that $B \setminus A \subseteq (\partial B)_{+\eta}$. To this end, we first prove the intermediate result $B \setminus A \subseteq A_{+\eta} \setminus A \subseteq (\partial A)_{+\eta}$.

We find that $B \subseteq A_{+\eta}$ follows from (4.8), implying $A_{+\eta} \setminus A \supseteq B \setminus A$. We then wish to show that $A_{+\eta} \setminus A \subseteq (\partial A)_{+\eta}$. Indeed, $A_{+\eta} \setminus A$ includes all points outside A that are at most distance η away from the nearest point in A. Any line from a point exterior to A into A must first pass through ∂A, which implies that for all $x \notin A$, $d(x,A) = d(x,\partial A)$ [28, p. 109, Lm. 3]. In other words, all points in $A_{+\eta} \setminus A$ are included in $(\partial A)_{+\eta}$.

We are now ready to prove $B \setminus A \subseteq (\partial B)_{+\eta}$. Take any $x \in B \setminus A$. If $x \in \partial B$, it is clearly in $(\partial B)_{+\eta}$; let us therefore take $x \in B \setminus A$ and $x \notin \partial B$. Because $B \setminus A \subseteq (\partial A)_{+\eta}$, there exists $y \in \partial A$ such that $d(x,y) \leq \eta$. In fact, let us take $y = \arg\min_{y\in\partial A} d(x,y) = \arg\min_{y\in A} d(x,y)$, which exists by the compactness of A; recall that the second equality follows from the argument that any line from any point exterior to A into A must first pass through ∂A. Since $x \notin A$ and A is compact, we find $x \neq y$. Consider a ray starting at y and passing through x. Because of the compactness of B, this ray will pass through some $z \in \partial B$, with $x \neq z$. It suffices to show $d(x,z) \leq \eta$ to prove our claim.

We know that $d(x,z) \leq d(y,z)$, so it suffices to show $d(y,z) \leq \eta$. Assume by contradiction that $d(y,z) > \eta$. Then because $z \in \partial B$ and $\partial B \subseteq (\partial A)_{+\eta}$, there exists $q \in \partial A$ such that $d(z,q) < d(y,z)$ with $y \neq q$. We note that $x = \lambda y + (1-\lambda)z$ for some $\lambda \in (0,1)$. Hence

$$d(x, \lambda y + (1-\lambda)q) \leq (1-\lambda)d(z,q) < (1-\lambda)d(y,z) = d(x,y).$$

Because of convexity of A, we have $\lambda y + (1-\lambda)q \in A$, which contradicts the assumption that y is the closest point in A to x. Thus we showed $d(y,z) \leq \eta$, which gives $d(x,z) \leq \eta$, and therefore $x \in (\partial B)_{+\eta}$. This proves $B \setminus A \subseteq (\partial B)_{+\eta}$.

Let us now define $C := B \setminus (\partial B)_{+\eta}$. We obtain

$$B \setminus A \subseteq (\partial B)_{+\eta} \cap B = B \setminus [B \setminus (\partial B)_{+\eta}] = B \setminus C,$$

where we utilized a set identity that expresses an intersection in terms of complements (see, e.g., [18, p. 30, Thm. 2.19(ii)]). From this last inclusion, given that $C \subseteq B$, we obtain $C \subseteq A$, which completes the proof. □

In Thm. 4.1(v), the quality of the inner approximation strongly depends on the quality of the trajectory deviation growth bound used in Cor. 3.1. In practice, the inner approximation is tight for short times from t_0 as shown in the next section, but it is often necessary to reinitialize the inequality and underlying nominal reachable set when the time difference becomes too large.

Thm. 4.1 relies on a number of conditions, including the shared R'-convexity of the multifunctions F and F_ϵ, that may be difficult to verify for a general nonlinear system. As mentioned, our approach also seems to work for systems that do not satisfy these technical assumptions as demonstrated in Sec. 5.2. Easing these assumptions is a subject of our future work.

The following proposition shows that, for systems affine in control, i.e., given by $\dot{x} = f(t,x,u) = f_x(t,x) + f_u(t,x)u$, with $f_x : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $f_u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$, the shared R'-convexity condition holds under some assumptions on $f_u(t,x)$ for $m \geq n$. Affine-in-control dynamics see widespread use in system modeling and control applications, e.g., in aerospace vehicles [12, §12.4] and many other mechanical systems [30].

PROPOSITION 4.1. (SHARED R'-CONVEXITY) *Given $R \geq 0$, let both $\mathscr{U} \subset \mathbb{R}^m$ and $\mathscr{U}_\epsilon \subseteq \mathscr{U}$ be R-convex. Consider $f : [t_0,\infty) \times \mathbb{R}^n \times \mathscr{U} \to \mathbb{R}^n$ given by $f(t,x,u) = f_x(t,x) + f_u(t,x)u$, where $m \geq n$. Let $f_u(t,x)$ have full rank for all $(t,x) \in \mathbb{R} \times \mathbb{R}^n$.*

Then $F(t,x) := f(t,x,\mathscr{U})$ and $F_\epsilon(t,x) := f(t,x,\mathscr{U}_\epsilon)$ are R'-convex, with

$$R' := R \sup_{(t,x)\in[t_0,\infty)\times\mathbb{R}^n} \frac{\lambda_{\max}(f_u(t,x)f_u^\mathsf{T}(t,x))}{\sqrt{\lambda_{\min}(f_u(t,x)f_u^\mathsf{T}(t,x))}},$$

where $\lambda_{\max}$ and $\lambda_{\min}$ denote the minimum and maximum eigenvalue of their argument, respectively.

Proof. Given the premise, the proof follows by application of [23, p. 264, Cor. 1] and [23, p. 267, Cor. 9]. □

In the next section we proceed to illustrate our theory by way of two numerical examples.

5 Numerical Examples.

We consider two numerical examples: a control system with quadratic nonlinearities and aircraft wing rock dynamics. We have chosen low-order systems for ease of exposition; computing an analytical bound for higher dimensional systems is possible, albeit more cumbersome. We will show how both Cor. 3.1 and Thm. 4.1 can be applied to these systems. For both examples, we have used the CORA MATLAB toolkit [2] to compute the nominal and off-nominal reachable sets. In practice, the nominal reachable set would be computed prior to the system's operation using a similar toolkit. The methods used in such toolkits can, however, often not be used online because of hardware limitations and poor scalability, hence the need for an approach such as ours.

5.1 Illustrative System with Quadratic Nonlinearities. Consider the following illustrative affine-in-control system with quadratic nonlinearities, used in [8, p. 70, Sec. 1.3]:

$$(5.9)\quad \dot{x}(t) = f(x(t),u(t)) = \begin{bmatrix} 2x_1 \\ 2x_2 + 4x_1^2 + x_2^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u.$$

Note that [8] sought to inner approximate the forward reachable tube of dynamics (5.9), which is the union of forward reachable sets over a time interval; this means that our results cannot be compared directly with those of [8]. In this example, we define multifunction F as $F(x) := f(x,\mathscr{U})$, with $\mathscr{U} = \mathcal{B}_{0.1}$ and the impaired control set is $\bar{\mathscr{U}} = \mathcal{B}_{0.08}$, with $\epsilon = d_\mathrm{H}(\mathscr{U},\bar{\mathscr{U}}) = 0.02$. Note that we only require an upper bound on $d_\mathrm{H}(\mathscr{U},\bar{\mathscr{U}})$ to apply our theory. We take initial set of states to be $\mathscr{X}_0 = \mathcal{B}_{0.1}$. In this example, it is evident that Prop. 4.1 is applicable, and both $F(t,x)$ and $\bar{F}(t,x)$ are R'-convex with $R' = 0.1$, since both $\mathscr{U}$ and $\bar{\mathscr{U}}$ are R-convex with $R = 0.1$.

We can easily find the following bound on trajectory deviation growth:

$$(5.10)\quad \begin{aligned} \|\tilde{f}(\bar{x},\bar{u})\| &= \|f(x+\tilde{x},u+\tilde{u}) - f(x,u)\| \\ &\le (4+10M)\|\tilde{x}\| + 5\|\tilde{x}\|^2 + 2\|\tilde{u}\|, \end{aligned}$$

where $M := \max_{y\in\mathbb{X}_t^{\rightarrow}(F,\mathscr{X}_0)} \|y\|$. The above inequality follows straightforwardly from the triangle inequality. Since the off-nominal reachable set is guaranteed to be a subset of the nominal set by Thm. 4.1(iii), M in (5.10) will be an upper bound for the norm of elements in the off-nominal reachable set as well.

Since the bound on $\|\bar{f}(\bar{x},\bar{u})\|$ is strictly increasing in $\|\bar{x}\|$ for all $\|\bar{u}\|$, we can apply Cor. 3.1. Bound $\eta(t,\epsilon)$ can be straightforwardly found by numerical means. We have used a combination of Runge–Kutta numerical integration, and Newton's method to evaluate G^{-1}, yielding a computationally efficient way of obtaining $\eta(t,\epsilon)$ for use in Thm. 4.1(v). Comparing this to the symbolic Taylor expansions and abstraction error computations required for the computation of the reachable set in [2], our numerics only rely on simple floating point operations and function evaluations, agnostic of system order. We evaluate the inner approximations at times $t \in \{0.1, 0.25, 0.4\}$, yielding the results shown in Fig. 1.

To assess the performance of the inner approximation, we compare the ratio between the volume of the inner approximated set and the off-nominal set. This gives 96% at $t = 0.1$, 83% at $t = 0.25$, and 29% at $t = 0.4$. As expected, the conservative bound on the trajectory deviation growth causes the inner approximation to shrink over time. Nevertheless, at times close to the initial time, the inner approximation is remarkably close to, yet guaranteed to be a subset of, the off-nominal reachable set.

5.2 Wing Rock. We now raise an example that is relevant for delta-wing aircraft flying at high angles of attack, where the aircraft experiences wing rock [3]. This phenomenon causes the aircraft to roll due to flow asymmetries that arise as a result of the high angle of incidence of the wing with respect to the airflow, causing the aircraft to experience nonlinear roll damping and limit cycle oscillations [7]. We consider the following simplified nonlinear wing rock model [7]:

$$\begin{aligned} f(x,u) = \dot{x} &= \begin{bmatrix} \dot{\phi} \\ \dot{p} \end{bmatrix} \\ &= \begin{bmatrix} p \\ \theta_1\phi + \theta_2 p + (\theta_3|\phi| + \theta_4|p|)p + \theta_5\phi^3 \end{bmatrix} + \begin{bmatrix} 0 \\ \theta_6 \end{bmatrix} u, \end{aligned}$$

where ϕ and p denote the roll angle and roll rate, respectively, and u denotes the aileron deflection. The coefficients θ are taken as in [7]: $\theta = \begin{bmatrix} -0.018 & 0.015 & -0.062 & 0.009 & 0.021 & 0.75 \end{bmatrix}^\mathsf{T}$. Note that in this example, Prop. 4.1 does not hold, and we cannot make claims about the R-convexity of $F(t,x)$ and $\bar{F}(t,x)$; this example serves to show that our theory can also extend to these relaxed cases, although a formal investigation is left as future work.

Here, we take $\mathscr{U} = [-0.175, 0.175]$ and $\bar{\mathscr{U}} = [-0.1663, 0.1488]$, yielding $\epsilon = 0.0263$. The nominal con-

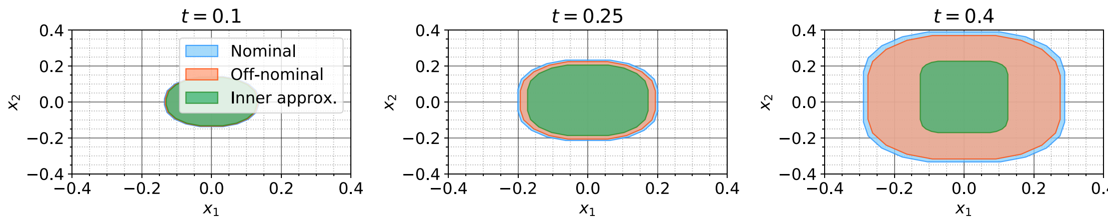

Figure 1: Nominal and off-nominal reachable sets, as well as the computed inner approximation of the reachable set, for the example in Sec. 5.1.

trol inputs correspond to aileron deflections of $[-10, 10]$ degrees, while the off-nominal control inputs incorporate a 15% decrease in stick-forward aileron authority, and a 5% decrease in stick-backward authority. The initial set of states is taken to be $\mathcal{X}_0 = [0.2, 0.25]^2$ to capture a starboard roll at high roll rate, and the reachable set is evaluated at times $t \in \{0.1, 0.5, 0.75\}$.

Deriving the trajectory deviation growth bound follows a similar approach as in the previous example, with elementary applications of the triangle inequality; the steps are omitted here. We obtain

$$\|\bar{f}(\bar{x}, \bar{u})\| \leq \|\bar{x}\| \left[1 + |\theta_1| + |\theta_2| + (|\theta_3| + |\theta_4|)(2M + \|\bar{x}\|)\right] + |\theta_5|\|\bar{x}\|^3 + |\theta_6|\|\bar{u}\|,$$

where once again we have $M = \max_{y \in \mathbb{X}_t^{\rightarrow}(F, \mathcal{X}_0)} \|y\|$.

The reachable set inner approximations obtained for the wing rock model are shown in Fig. 2. In this case, the ratios between the volume of the inner approximated set and the off-nominal set decrease slower than in the first example: 92% at $t = 0.1$ s, 67% at $t = 0.5$ s, and 54% at $t = 0.75$ s. This example shows that adequate inner approximations can easily be obtained even for comparatively large time intervals, which should allow for sufficient time for maneuver planning considering that the time delay and maneuvering time for roll maneuvers in fighter aircraft adds up to between 1 to 2 seconds [17, p. 86]. In practice, these results could allow for aircraft to plan steep turns under degraded control authority, such as partial hydraulics failure of the aileron actuators or control authority degradation at high angles of aileron deflection due to flow separation, so as to not enter potentially unrecoverable regimes of wing rock.

6 Conclusion.

In this work, we have introduced a new technique for efficiently computing an inner approximation to a reachable set, in case of diminished control authority, given basic knowledge of the trajectory deviation growth as well as a precomputed nominal reachable set. We have shown that the ability to compute these approximations online can have practical application to control of dynamical systems in off-nominal conditions. To obtain an inner approximation of the reachable set under diminished control authority, we have given an integral inequality that provides an upper bound on the minimal trajectory deviation between the nominal and off-nominal systems. Our approach uses this upper bound on the minimal trajectory deviation to compute an inner approximation of the off-nominal reachable set based on the nominal set. These results can be applied online on systems at a low computational cost.

We have demonstrated our approach by two numerical examples: an illustrative control-affine system with quadratic nonlinearities and a wing rock model. The numerical examples indicate that periodic reinitialization of the reachable set is required to ensure tightly bounding inner approximations, with the tightness of the bound being strongly related to the quality of the trajectory deviation bound.

Going forward, we intend to extend the results of this work to support bounded changes in off-nominal dynamics, which will require inner approximations of the interval of existence of a convex reachable set for the off-nominal system. Additionally, we plan to investigate the applicability of our results to multifunctions that only guarantee convexity, and not R-convexity, such as the example in Sec. 5.2. Along with this effort, we will develop a control method that drives the system to a state in the computed inner approximation of the off-nominal reachable set. One possible avenue for this is by expressing the inner approximated reachable set as a polytopic state constraint, and employing model predictive control (MPC) to drive the system to this set under the diminished control input constraints [6, p. 182].

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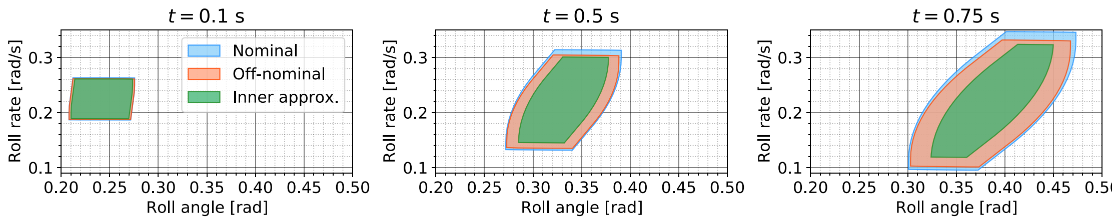

Figure 2: Nominal and off-nominal reachable sets, and inner approximation of the reachable set, for the wing rock dynamics.

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