Robustly Linearized Model Predictive Control for Nonlinear Infinite-Dimensional Systems ⋆

Hamza El-Kebir * **Richard Berlin** ** **Joseph Bentsman** *** **Melkior Ornik** ****

* *Dept. of Aerospace Engr., Urbana, IL 61801, USA (e-mail: elkebir2@illinois.edu)*
** *Dept. of Trauma Surgery, Carle Hospital Dept. of Biomedical and Translational Sciences, Carle Illinois College of Medicine, Urbana, IL 61801, USA*
*** *Dept. of Mech. Sci. and Engr., Urbana, IL 61801, USA (e-mail: jbentsma@illinois.edu)*
**** *Dept. of Aerospace Engr., Urbana, IL 61801 (e-mail: mornik@illinois.edu)*

Abstract: This work presents a computationally efficient approach to robustly linearized model predictive control for nonlinear affine-in-control evolution equations on infinite-dimensional system state. In this setting, robust linearization refers to a need to account for the approximation errors in linearization and discretization in the model predictive control law, such that the original output constraints are not violated on the true system, a feature that present model predictive control frameworks lack. The main purpose of this work is to enable tractable model predictive control for nonlinear distributed parameter systems while accounting for these approximation errors by means of output constraints. These output constraints are derived using tight integral inequalities that rest on mild assumptions on the nonlinear system dynamics, and are easy to evaluate in real-time. Using our method, linearization and discretization errors are explicitly accounted for, producing for the first time a model predictive control law that is robust to approximation errors. This approach hence enables a trade-off between computational efficiency and strictness of output constraints, much akin to robust control methods. We demonstrate our method on a nonlinear distributed parameter system, namely a one-dimensional heat equation with a velocity-controlled moveable heat source, motivated by autonomous energy-based surgery.

Keywords: Model predictive control for distributed parameter systems, uncertain systems, constrained control.

1. INTRODUCTION

The notion of optimal control for infinite-dimensional or distributed parameter systems has received considerable attention in the literature (Dubljevic et al., 2006; Humaloja and Dubljevic, 2018; Dubljevic and Humaloja, 2020), most often in the context of linear systems. A common approach is to discretize such an infinite-dimensional linear system in time and space, thereby obtaining an approximate discrete-time spatially discretized formulation of the system dynamics that is directly amenable to optimal control methods for linear systems such as model predictive control (MPC) with quadratic cost. While model order reduction of this form produces tractably solvable control problems, there exist no general guarantees that the infinite-dimensional continuous-time system will converge to the control objective, or even be stable, when subjected to such a reduced-order control law. Some past work has aimed to derive a class of finite-dimensional controllers for infinite-dimensional linear systems that are robust under bounded changes in the dynamics (Paunonen and Phan, 2020), thereby providing a possible avenue for accounting for discretization errors. However, this line of work imposes strong assumptions on the type of model order reduction, namely Galerkin approximations, which may be prohibitive to abide by in practice and intractable to implement for general systems. The need for spatial discretization was first lifted by Humaloja and Dubljevic (2018); Dubljevic and Humaloja (2020), where the Cayley–Tustin transform was used for time discretization, and an MPC problem was formulated by penalizing the finite-dimensional output and input. Following results from (Havu and Malinen, 2007), a stabilizing control law for the discrete-time system was shown to also stabilize its continuous-time counterpart, thus allowing for classical results in MPC theory to be leveraged directly. However, the resulting method is only applicable to linear systems,

⋆ This work was supported by NASA grant no. 80NSSC21K1030 and NASA ULI project "Robust and Resilient Autonomy for Advanced Air Mobility," as well as the National Institute of Biomedical Imaging and Bioengineering of the National Institutes of Health under award number R01EB029766.

and may be intractable to implement because of a reliance on resolvent operators for infinite-dimensional systems. In Toledo et al. (2020) and Wu et al. (2021), methods for reduced-order observer and controller design are presented for Port–Hamiltonian systems, based on the assumption that an approximate reduced-order dynamical model is available; however, these approaches are limited to linear systems and hinge on the key assumption of the availability of a controllable reduced-order approximation of the PDE.

In this work, we formulate a novel model predictive control method for *nonlinear* infinite-dimensional systems, termed as *robustly linearized model predictive control* (RoLiMPC), which greatly simplifies the implementation process by introducing an *approximate Cayley–Tustin transform* in which the resolvent operator is approximated by a truncated series expansion. Most importantly, our method is robust to linearization *and* discretization errors by means of analytical error bounds that are used to tighten the output constraints, such that the control law is guaranteed to honor the original output constraints when applied to the original nonlinear system. Doing so presents a major improvement over prior work, where the only convergence proofs provided are asymptotic in the discretization time step (Dubljevic and Humaloja, 2020), and only small nonlinearities may be captured (Paunonen and Phan, 2020). The approximation error bounds in our work are based on a combination of a truncated Neumann series approximation of the resolvent operator, as well as a modified version of an integral inequality known as the Bihari inequality based on mild assumptions on the regularity of the system dynamics (El-Kebir and Ornik, 2021; El-Kebir et al., 2022a). The resulting optimization problem that solves for the optimal control signal is finite-dimensional and tractable, requiring only the derivation of the explicit adjoint operators belonging to the linearized system dynamics.

Note that formal proofs have been omitted due to space constraints; proofs will be provided in a future publication.

2. PRELIMINARIES

Let $\mathcal{L}(X,Y)$ denote the set of bounded linear operators between normed spaces X and Y. For any two $x, x' \in X$, let $[x, x']$ denote the set of all convex combinations between x and x'. For a bounded linear operator $T \in \mathcal{L}(X,Y)$, let $T^* \in \mathcal{L}(Y,X)$ denote its adjoint. For a set $K \subseteq X$, let χ_K denote is characteristic function. Let $H^k(X)$ denote the Sobolev space of order k defined by the L^2-norm. Let $\mathcal{D}(A)$, $\mathcal{R}(A)$, $\mathcal{N}(A)$, $\sigma(A)$, $\rho(A) := \mathbb{C} \setminus \sigma(A)$ denote the domain, range, kernel, spectrum, and resolvent of A, respectively. For some $x \in X \ni 0$ and $r \in \mathbb{R}_+ := \{a \in \mathbb{R} : a \geq 0\}$, let $\mathcal{B}(r,x)$ denote a closed ball of radius r centered at x; if $x = 0$, we write $\mathcal{B}(r) := \mathcal{B}(r,0)$. We denote by δ_ξ the Dirac delta function centered at ξ and by φ_ξ the evaluation map $\varphi_\xi f = f(\xi)$ for some function f with $\xi \in \mathcal{D}(f)$. For a linear operator $A : \mathcal{D}(A) \subseteq X \to X$ and a fixed $\lambda_0 \in \rho(A)$, we define the scale spaces (Tucsnak and Weiss, 2009, p. 59, Sec. 2.10)

$$X_1 := (\mathcal{D}(A), \|(\lambda_0 - A)\ \cdot \|), \tag{1}$$

$$X_{-1} := \overline{(X, \|(\lambda_0 - A)^{-1}\ \cdot \|)}, \tag{2}$$

which satisfy $X_1 \subseteq X \subseteq X_{-1}$. Let $A_{-1} \in \mathcal{L}(X, X_{-1})$ denote the Yosida extension of $A \in \mathcal{L}(X,X)$ to X_{-1}, i.e., the unique linear extension of A to $\mathcal{L}(X, X_{-1})$ (Weiss, 1994, p. 26). For ease of exposition, we will write for any $s \in \mathbb{C}$, $(s - A) = (sI - A)$, understanding that I is the identity operator defined in the appropriate space.

We briefly recount the notion of *Fréchet differentiability* (Munkres, 1991, Sec. 5, p. 41).

Definition 1 (Fréchet Differentiability)**.** A function $f : U \subseteq X \to Y$, where X and Y are normed vector spaces, and U is open, is *Fréchet differentiable at* $x \in U$ if there exists $A \in \mathcal{L}(X,Y)$ such that:

$$\lim_{\|h\| \to 0} \frac{\|f(x+h) - f(x) - Ah\|_Y}{\|h\|_X} = 0. \tag{3}$$

We may now define the robustly linearized model predictive control problem. For the sake of brevity, we consider in this work nonlinear affine-in-control *regular nonlinear systems* (Natarajan and Bentsman, 2016), i.e., systems that produce regular linear systems (Weiss, 1994) when linearized. The robustly linearized model predictive control problem is formalized as follows:

Problem 1. Consider a nonlinear affine-in-control dynamical system of the form

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + g(x(t))u(t), \quad x(0) = x_0, \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{4}$$

where $f : \mathcal{D}(f) \subseteq X \to X$ and $g \in \mathcal{D}(g) \subseteq U \to X$ are Fréchet differentiable, $C \in \mathcal{L}(X,Y)$, and $D \in \mathcal{L}(U,Y)$. X, Y, U are assumed to be separable Hilbert spaces. In addition, assume $\dim Y < \infty$ and $\dim U < \infty$, with $U \subseteq \mathcal{B}(M_u)$ where $\mathcal{B}(M_u)$ is a closed ball with known radius $M_u > 0$.

Across a time horizon $T > 0$, given some sampling period $h \in (0, T]$ such that $T/h =: N \in \mathbb{N}$, produce a control signal $u^* : [0,T] \to U$ that solves a *linear* finite-horizon model predictive control problem:

$$\begin{aligned} (u^*[k])_{k=0}^{N-1} = \arg \min_{(u[k])_{k=0}^{N-1}} \sum_{k=0}^{N-1} \langle y[k], Qy[k] \rangle + \langle u[k], Ru[k] \rangle \\ + \langle y[N], Sy[N] \rangle, \end{aligned} \tag{5}$$

subject to

$$u[i] \in \bar{U}_i \subseteq U, \quad y[i] \in \bar{Y}_i \subseteq Y \tag{6}$$

for all $i = 0, \ldots, N-1$, where $\bar{U}_i$ and $\bar{Y}_i$ are compact convex sets, and Q, R, S are positive self-adjoint operators. $u^*(t)$ is the zero-order hold continuization of $(u^*[k])_{k=0}^{N-1}$. Here, $[k] = kh$, and $y(t)$ is given by a *linearized and discretized approximation* of (4) such that (5) is a quadratic program.

In order to tractably solve Problem 1, we make the following mild assumption on the regularity of the Fréchet derivatives of f and g in (4):

Assumption 1. For $x, x' \in X$, the Fréchet derivative on f satisfies the following inequality:

$$\|\mathbf{D}^2 f(x) - \mathbf{D}^2 f(x')\| \leq w_f(\|x - x'\|), \tag{7}$$

and g satisfies:

$$\|\mathbf{D} g(x) - \mathbf{D} g(x')\| \leq w_g(\|x - x'\|), \tag{8}$$

where $w_f : \mathbb{R}_+ \to \mathbb{R}_+$ and $w_g : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous, monotonic, nondecreasing, convex functions.

In order to solve Problem 1, we broadly trace the following steps: (1) account for truncation error for resolvent operator, yielding the discretization error; (2) account for linearization error using an overapproximate differential inclusion, and use this error to construct a bound on the linearization error using the Bihari inequality; (3) tighten the original MPC output constraints by the sum of the discretization and linearization error bounds; (4) solve constrained LMPC using a modified version of the method by Dubljevic and Humaloja (2020).

In the remainder of this work, we will make use of the Neumann series expansion to approximate the resolvent operator of a bounded linear functional.

Lemma 1 (Neumann Series). *Let $A \in \mathcal{L}(X)$, such that $\|A\| < 1$. Let $R(\lambda, A) : \mathbb{C} \times \mathcal{L}(X) \to \mathcal{L}(X)$ denote the resolvent operator, which is defined by*

$$R(\lambda, A) := (\lambda - A)^{-1}. \tag{9}$$

Then, the resolvent operator is characterized by the following convergent Neumann series:

$$R(\lambda, A) = \sum_{i=0}^{\infty} A^n. \tag{10}$$

A key tool in producing nonlinear integral inequalities that will be leveraged extensively is the Bihari inequality, extended to the controlled case by El-Kebir and Ornik (2021).

Theorem 1 (Extended Bihari inequality (El-Kebir and Ornik, 2021, Theorem 3.1)). *Let $x(t)$ be a solution to the equation*

$$\dot{x} = h(t, x, u), \quad x(t_0) = x_0, \quad 0 \leq t_0 \leq t < \infty,$$

where $h(t, x, u) : [t_0, \infty) \times X \times U \to X$ is continuous for $t_0 \leq t < \infty$, and U is compact and finite-dimensional and satisfies $\max_{u \in U} \|u\| = M_u$. Assume that

$$\|h(t, x, u)\| \leq a(t) w(\|x\|, \|u\|) + b(t),$$

for all $x \in X$, $u \in U$, $t_0 \leq t < \infty$, where a, b are continuous and positive and w is continuous, monotonic, nondecreasing and positive. In addition, w is uniformly monotonically nondecreasing in $\|u\|$.

Then,

$$\|x(t)\| \leq G^{-1} \left[G \left(\|x(t_0)\| + \int_{t_0}^{t} b(\tau) \mathrm{d}\tau \right) + \int_{t_0}^{t} a(\tau) \mathrm{d}\tau \right], \tag{11}$$

where the expression on the right-hand side is strictly increasing in t. In (11), we define

$$G(r) := \int_{r_0}^{r} \frac{\mathrm{d}s}{w(s, M_u)}, \quad r > 0, r_0 > 0,$$

for arbitrary $r_0 > 0$ and for all $t \geq t_0$ for which it holds that

$$G \left(\|x(t_0)\| + \int_{t_0}^{t} b(\tau) \mathrm{d}\tau \right) + \int_{t_0}^{t} a(\tau) \mathrm{d}\tau \in \mathrm{dom}(G^{-1}).$$

We now proceed by introducing the Cayley–Tustin method of time discretization, for which we produce approximations with tight analytical error bounds that are readily evaluated.

3. NONLINEAR CAYLEY–TUSTIN TIME DISCRETIZATION

We can now define the nonlinear Cayley–Tustin time discretization, and pose an error bound to account for the linearization error. Time discretization is instrumental in formulating tractable optimization problems, since the solution of a discretized finite-horizon optimization problem will be a finite-dimensional vector, as opposed to a function. In addition, the Cayley–Tustin discretization (Havu and Malinen, 2007) method allows us to forgo spatial discretization, which is a source of additional errors. In its original form, the Cayley–Tustin discretization scheme is only defined for linear systems, and requires the evaluation of a resolvent operator that is hard to evaluate in general. We propose a novel linearized approximate discretization scheme, for which we prove tight approximation error bounds that can readily be evaluated.

The Cayley–Tustin scheme required evaluation of the resolvent operator $R(\lambda, A) := (\lambda - A)^{-1}$, which is often prohibitive to obtain analytically except in the case of one-dimensional linear systems (Dubljevic and Humaloja, 2020). We therefore proceed by deriving a finite series approximation result for this operator with known sharp truncation error bounds:

Proposition 1 (Approximation of the Resolvent Operator). *Let $A \in \mathcal{L}(X)$, and let $\lambda \in \mathbb{R}_+ \setminus \{0\}$. If $\|A/\lambda\| := \sup_{\substack{x \in X \\ \|x\|=1}} \|Ax/\lambda\| < 1$, then*

$$R(\lambda, A) = \lambda^{-1} \sum_{i=0}^{\infty} (A/\lambda)^i, \tag{12}$$

which is a convergent Neumann series. Let $\epsilon \in (0, 1)$ be given. Then $\lambda \geq \frac{\|A\|}{1-\epsilon}$ ensures that the truncated Neumann series

$$R_N(\lambda, A) := \lambda^{-1} \sum_{i=0}^{N-1} (A/\lambda)^i \tag{13}$$

has truncation error

$$\|R(\lambda, A) - R_N(\lambda, A)\| \leq \lambda^{-1} \frac{(1-\epsilon)^N}{\epsilon}. \tag{14}$$

We may now bound the approximation error over discrete time steps due to truncation of the resolvent operator as follows.

Proposition 2 (Approximate Linearized Cayley–Tustin Discretization). *Considering the Cayley–Tustin discretization given by Havu and Malinen (2007) and the finite series approximation to the resolvent operator from Proposition 1, we find that for some given $N \in \mathbb{N}$ and $x_0 \in X$, the approximate linearized discrete system given by*

$$\begin{aligned} \Delta\hat{x}[k] &= \hat{A}_{x_0,\delta} \Delta\hat{x}[k-1] + \hat{B}_{x_0,\delta} u[k], \quad \hat{x}[0] = x_0 \\ y[k] &= \hat{C}_{x_0,\delta} \hat{x}[k-1] + \hat{D}_{x_0,\delta} u[k], \quad k \geq 1, \end{aligned} \tag{15}$$

where $\Delta\hat{x}[j] := \hat{x}[j] - \hat{x}[0]$ for all $j \geq 0$, and

$$\hat{A}_{x_0,\delta} = (\delta + A_{x_0}) R_N(\delta, A_{x_0}), \tag{16a}$$

$$\hat{B}_{x_0,\delta} = \sqrt{2\delta} R_N(\delta, A_{x_0}) B_{x_0}, \tag{16b}$$

$$\hat{C}_{x_0,\delta} = \sqrt{2\delta} C R_N(\delta, A_{x_0}), \tag{16c}$$

$$\hat{D}_{x_0,\delta} = \hat{\mathcal{G}}(\delta), \tag{16d}$$

where

$$\hat{\mathcal{G}}(s) := C R_N(s, A_{x_0}) B_{x_0} + D, \tag{17}$$

satisfies

$$\|\Delta x[k] - \Delta\hat{x}[k]\| \leq \|\hat{A}_{x_0,\delta}\|\|\Delta x[k-1] - \Delta\hat{x}[k-1]\| \\ + (1 + \|A_{x_0}\|/\delta)[(1-\epsilon)^N/\epsilon]\|\Delta\hat{x}[k-1]\| \\ + \sqrt{2/\delta}\|B_{x_0}\|[(1-\epsilon_{-1})^N/\epsilon_{-1}]\|u[k]\|, \tag{18}$$

for $k \in \mathbb{N}$, where $\epsilon, \epsilon_{-1} \in (0,1)$ are such that $\|A_{x_0}/\delta\| \leq 1-\epsilon$ and $\|(A_{x_0})_{-1}/\delta\| \leq 1-\epsilon_{-1}$.

We can expand the recurrence relation given in Proposition 2 to obtain a bound for the discretization error as shown next.

Corollary 1. Assuming $u[k] \in U \subseteq \mathcal{B}(M_u)$ for some $M_u \geq 0$, we have

$$\|\Delta x[k] - \Delta\hat{x}[k]\| \leq \alpha_1^k\|\Delta x[0] - \Delta\hat{x}[0]\| \\ + \sum_{i=0}^{k-1}\alpha_1^{k-1-i}\alpha_2\gamma[i] + \alpha_1^i\beta M_u =: \hat{\eta}[k], \tag{19}$$

where

$$\alpha_1 := \|\hat{A}_{x_0,\delta}\|,\ \alpha_2 := (1 + \|A_{x_0}\|/\delta)[(1-\epsilon)^N/\epsilon], \tag{20a}$$

$$\beta := \sqrt{2/\delta}\|B_{x_0}\|[(1-\epsilon)^N/\epsilon], \tag{20b}$$

$$\gamma[k] := \|\hat{A}_{x_0,\delta}\|^k\|\Delta\hat{x}[0]\| + \sum_{i=0}^{k-1}\|\hat{A}_{x_0,\delta}\|^i\|\hat{B}_{x_0,\delta}\|M_u \tag{20c}$$

where the definition of β follows from the fact that $\epsilon = \epsilon_{-1}$. For discretization time $h > 0$, we can define the continuous function $\hat{\eta}(t)$ as follows:

$$\hat{\eta}(t) := \begin{cases} 0, & t \leq 0, \\ \hat{\eta}[\lceil t/h \rceil], & \text{otherwise.} \end{cases} \tag{21}$$

Having obtained a means of approximating the resolvent operator with a tight upper bound on the operator norm error, we wish to study the error introduced by linearizing f and g.

4. ROBUSTLY LINEARIZED MODEL PREDICTIVE CONTROL

In this section we present the robustly linearized model predictive control method for nonlinear control-affine systems on Hilbert spaces. As in the classical model predictive control setting, we wish to obtain a control signal that achieves quadratically optimal performance, while adhering to output and control input constraints (see Problem 1). Given the prohibitiveness of nonlinear MPC in the case of general infinite-dimensional control systems, we have chosen to linearize the system. If this linearized *continuous-time* system were to be employed in an MPC setting, the resulting optimization problem would have the form of a generalized Hamilton–Jacobi–Isaacs equation, with the solution involving a partial differential equation over a Hilbert space. Since such an equation is prohibitive to solve even in the finite-dimensional case, we have chosen to discretize the linearized system *in time*, resulting in a finite-dimensional (convex) quadratic program that solves the underlying MPC problem.

In the process described above, we have accrued both linearization and discretization errors, which should be incorporated in the output constraints to achieve safe control. The first part of this section provides a closed form tight upper bound to the approximation error, which can be computed efficiently online. This upper bound will be used to tighten the original MPC constraints so as to achieve a notion of robustly linearized MPC, where robustness refers to robustness to approximation of the dynamics, i.e., linearization and discretization.

4.1 Approximation Error Bound

We first consider the *approximation error*, which consists of components due both to linearization and discretization. Given Assumption 1, we find the following guaranteed differential inclusion:

Proposition 3. *For system* (4) *satisfying Assumption 1, we obtain the following differential inclusion:*

$$\dot{x}(t) \in A_{x_0}\Delta x(t) + B_{x_0}u(t) \\ + f(x_0) + \mathcal{B}(w_f(\|\Delta x(t)\|), \|\mathbf{D}^2 f(x_0)\|)\|\Delta x(t)\|^2/2 \\ + \mathcal{B}(w_g(\|\Delta x(t)\|), \|\mathbf{D}g(x_0)\|)\|x(t)\|u(t). \tag{22}$$

From the inclusion of Proposition 3, we can now derive a bound on the error dynamics between system (4) and its linearized counterpart:

$$\begin{aligned} \Delta\dot{\bar{x}}(t) &= A_{x_0}\Delta\bar{x}(t) + B_{x_0}\bar{u}(t), \quad \Delta\bar{x}(0) = 0, \\ \Delta\bar{y}(t) &= C\Delta\bar{x}(t) + D\bar{u}(t). \end{aligned} \tag{23}$$

Proposition 4. *For system* (4) *satisfying Assumption 1, the error dynamics between its solution and that of the linearized system* (23) *for $u(t) = \bar{u}(t)$, defined as $\Delta\dot{\tilde{x}}(t)$ where $\Delta\tilde{x}(t) := \Delta x(t) - \Delta\bar{x}(t)$, is characterized as follows:*

$$\|\Delta\dot{\tilde{x}}(t)\| \leq \boldsymbol{a}^\mathsf{T}(t)\boldsymbol{w}(\|\Delta\tilde{x}(t)\|) + b(t), \tag{24}$$

where

$$\boldsymbol{a}(t) := \begin{bmatrix} 1 \\ \|\Delta\bar{x}(t)\| \\ \|\Delta\bar{x}(t)\|^2 \\ \|\bar{u}(t)\| \\ \|\bar{u}(t)\|\|\Delta\bar{x}(t)\| \\ w_f(\|\Delta\bar{x}(t)\|) \\ w_f(\|\Delta\bar{x}(t)\|)\|\Delta\bar{x}(t)\| \\ w_g(\|\Delta\bar{x}(t)\|)\|\bar{u}(t)\| \end{bmatrix},$$

$$\begin{aligned} \boldsymbol{w}_1(\|\Delta\tilde{x}(t)\|) &:= \|A_{x_0}\|\|\Delta\tilde{x}(t)\| + [\|\mathbf{D}^2 f(x_0)\| \\ &\quad + w_f(\|\Delta\tilde{x}(t)\|)]\|\Delta\tilde{x}(t)\|^2/2, \\ \boldsymbol{w}_2(\|\Delta\tilde{x}(t)\|) &:= [\|\mathbf{D}^2 f(x_0)\| + w_f(\|\Delta\tilde{x}(t)\|)]\|\Delta\tilde{x}(t)\|, \\ \boldsymbol{w}_3(\|\Delta\tilde{x}(t)\|) &:= \|\mathbf{D}^2 f(x_0)\|/2 + w_f(\|\Delta\tilde{x}(t)\|)/2, \\ \boldsymbol{w}_4(\|\Delta\tilde{x}(t)\|) &:= [\|\mathbf{D}g(x_0)\| + w_g(\|\Delta\tilde{x}(t)\|)]\|\Delta\tilde{x}(t)\|, \\ \boldsymbol{w}_5(\|\Delta\tilde{x}(t)\|) &:= \|\mathbf{D}g(x_0)\| + w_g(\|\Delta\tilde{x}(t)\|), \\ \boldsymbol{w}_6(\|\Delta\tilde{x}(t)\|) &:= \|\Delta\tilde{x}(t)\|^2/2, \\ \boldsymbol{w}_7(\|\Delta\tilde{x}(t)\|) &:= \|\Delta\tilde{x}(t)\|, \\ \boldsymbol{w}_8(\|\Delta\tilde{x}(t)\|) &:= \|\Delta\tilde{x}(t)\|, \end{aligned}$$

and

$$b(t) := \|f(x_0)\| + [\|\mathbf{D}^2 f(x_0)\| + w_f(\|\Delta\bar{x}(t)\|)]\|\Delta\bar{x}(t)\|^2/2 \\ + [\|\mathbf{D}g(x_0)\| + w_g(\|\Delta\bar{x}(t)\|)\|]\|\bar{u}(t)\|\|\Delta\bar{x}(t)\|.$$

We can now explicitly formulate a possible linearization error bound.

Corollary 2. From (24), it is straightforward to see that all hypotheses of Thm. 1 are satisfied. If $\|\bar{x}(t)\|$ and $\|\bar{u}(t)\|$ are both known, this yields an inequality of the form

$$\|\Delta\tilde{x}(t)\| \leq \bar{\eta}(t),$$

where

$$\bar{\eta}(t) := G^{-1}\left[G\left(\|\Delta\tilde{x}(t_0)\| + \int_{t_0}^{t} b(\tau)\mathrm{d}\tau\right) + \int_{t_0}^{t}\|\boldsymbol{a}(\tau)\|\mathrm{d}\tau\right], \tag{25}$$

and

$$G(r) := \int_{r_0}^{r}\frac{\mathrm{d}s}{\|\boldsymbol{w}(s)\|}, \quad r > 0, r_0 > 0. \tag{26}$$

The discretization error has been derived in Corollary 1. Given the linearization and discretization errors, we may now present an upper bound for the total approximation error.

Theorem 2. *The output error due to linearization and discretization truncation errors is tightly bounded as follows*

$$\|\Delta\tilde{x}(t)\| \leq \bar{\eta}(t) + \hat{\eta}(t) =: \tilde{\eta}(t), \tag{27}$$

where $\bar{\eta}$ is given as in Corollary 2 and $\hat{\eta}$ is defined in (21).

Proposition 5. *For $N \to \infty$ and $h \to 0$ (equivalently, $\delta \to \infty$), the approximation error $\tilde{\eta}$ converges to zero.*

4.2 Constrained Optimization Problem

Based on the approximation error bound obtained in the previous subsection, we can now pose the main constrained optimization problem that underpins the robustly linearized model predictive control framework. The resulting model predictive control algorithm is robust to approximation errors due to linearization and discretization, as shown in Theorem 2. In this section we will omit the x_0 subscript for ease of exposition.

Theorem 3 (Robustly Linearized Model Predictive Control)**.** *Considering Problem 1 and a given horizon length N, resolvent truncation length N_r, and operating point $x_0 \in X$, assume that $\bar{U} \subseteq U$ and $\bar{Y} \subseteq Y$ are hyperrectangular axis-aligned sets, i.e.,*

$$(u[k])_{k=0}^{N-1} \in [\boldsymbol{u}_{\min}, \boldsymbol{u}_{\max}], \; (y[k])_{k=0}^{N-1} \in [\boldsymbol{y}_{\min}, \boldsymbol{y}_{\max}], \tag{28}$$

where $\boldsymbol{u}_{\min} := (\boldsymbol{u}_{\min,k})_{k=1}^{N} \in U^N$ and $\boldsymbol{y}_{\min} := (\boldsymbol{y}_{\min,k})_{k=1}^{N} \in Y^N$ are given; $\boldsymbol{u}_{\max}$ and $\boldsymbol{y}_{\max}$ are defined similarly. Let $m := \mathrm{dim} Y$.

An approximate solution to Problem 1 that honors the constraints of (28) *is obtained by solving the following constrained finite-dimensional quadratic program:*

$$\min_{\boldsymbol{u}\in U^N} J(\boldsymbol{u}, x_0) = \boldsymbol{u}^\mathsf{T} H\boldsymbol{u} + 2\boldsymbol{u}^\mathsf{T} P x_0, \tag{29}$$

subject to

$$\begin{bmatrix} I \\ -I \\ F \\ -F \end{bmatrix}\boldsymbol{u} \leq \begin{bmatrix} \boldsymbol{u}_{\max} \\ \boldsymbol{u}_{\min} \\ \max \boldsymbol{y}_{\max} - Gx_0 - ((\tilde{\eta}[k])_{i=1}^{m})_{k=1}^{N} \\ \min -\boldsymbol{y}_{\min} + Gx_0 + ((\tilde{\eta}[k])_{i=1}^{m})_{k=1}^{N} \end{bmatrix}, \tag{30}$$

where $H \in \mathcal{L}(U^N)$ is positive and self-adjoint, where

$$H_{i,j} := \begin{cases} \hat{D}_\delta^* Q \hat{D}_\delta + \hat{B}_\delta^* S \hat{B}_\delta + R, & i = j, \\ \hat{D}_\delta^* Q \hat{C}_\delta \hat{A}_\delta^{i-j-1}\hat{B}_\delta + \hat{B}_\delta^* S \hat{A}_\delta^{i-j}\hat{B}_\delta + R, & i > j, \\ H_{j,i}^*, & i < j, \end{cases} \tag{31}$$

and $P \in \mathcal{L}(X, U^N)$ is defined as $P_{i,\cdot} := \hat{D}_\delta^* Q \hat{C}_\delta \hat{A}_\delta^{i-1} + \hat{B}_\delta^* S \hat{A}_\delta^{i}$. The constraints are defined by

$$F_{i,j} := \begin{cases} \hat{D}_\delta, & i = j, \\ \hat{C}_\delta \hat{A}_\delta^{i-j-1}\hat{B}_\delta, & i > j, \\ 0, & i < j, \end{cases} \tag{32}$$

and $G \in \mathcal{L}(X, Y^N)$, with $G_{i,\cdot} := \hat{C}_\delta \hat{A}_\delta^{i-1}$. Here, $\tilde{\eta}$ is defined as in Thm. 2.

We refer the reader to Humaloja and Dubljevic (2018) (Sec. III.B) for an explicit example on how to obtain the adjoint operators that appear in Thm. 3. Given expressions for the adjoint operators, the resulting MPC algorithm can be implemented efficiently without explicit derivation of the resolvent operator, which necessitates numerical integration or series expansions in previous works (Humaloja and Dubljevic, 2018; Dubljevic and Humaloja, 2020), introducing errors that are not accounted for in the literature. Using our method, the resolvent operator can be obtained explicitly without the need for numerical integration, while the truncation error is explicitly accounted for by tightening the output constraints.

5. APPLICATION

In this work, we consider as an example the following thermodynamics model on a one-dimensional compact domain $\Omega \subseteq \mathbb{R}$, based on a damage-conscious tissue thermodynamics model (El-Kebir et al., 2022b):

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{\alpha}(t) \\ \dot{p}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} a\nabla^2 x(t) \\ \varrho^{x^* I - x(t)} \\ 0 \end{bmatrix}}_{f(\boldsymbol{x}(t))} + \underbrace{\begin{bmatrix} T_{p(t)} q & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}}_{g(\boldsymbol{x}(t))}\boldsymbol{u}(t), \tag{33}$$

where $x(t) \in H^1(\Omega)$ denotes the temperature field, $\alpha(t) \in H^2(\Omega)$ denotes the damage field, $p(t) \in \mathbb{R}$ denotes the heat source position, $a > 0$ refers to the thermal diffusivity, $q \in H^1(\Omega)$ denotes a unit heat source field, $\varrho \in (0, 1)$ is a coefficient that models the rate of damage accumulation, and $x^* \in \mathbb{R}_+$ is a critical damage temperature. The functional T_p is defined by

$$\begin{aligned} T_p &: H^1(\Omega) \to H^1(\Omega), \quad \forall p \in \Omega, \\ &T_p q(\eta) \mapsto \chi_\Omega(\eta) q(\eta - p), \quad \forall \eta \in \Omega, q \in H^1(\Omega). \end{aligned} \tag{34}$$

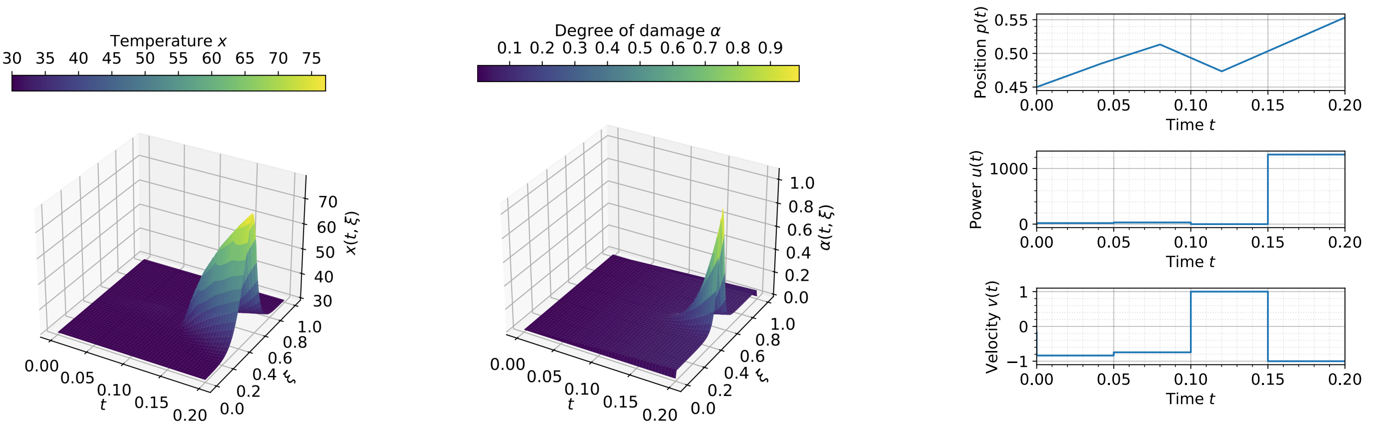
System (33) is of the form of (4) and belongs to the class of regular nonlinear systems, since its linearization at any point in the problem domain can be shown to be a regular linear system.

The full state is given as $\boldsymbol{x}(t) := (x(t), \alpha(t), p(t)) \in H^1(\Omega) \times H^2(\Omega) \times \mathbb{R}$, and the control inputs are denoted by $\boldsymbol{u}(t) := (u(t), v(t)) \in U \subseteq \mathbb{R}_+ \times \mathbb{R}$, where $u(t)$ denotes the nonnegative probe power and $v(t)$ refers to the probe velocity.

In this example we consider $\Omega = [0, 1]$, $a = 0.1$, $\varrho = 0.9$, $x^* = 40$, and $q = \chi_{[-\kappa,\kappa]}$, with $\kappa = 0.05$. We enforce in this problem the homogeneous Dirichlet boundary conditions $x(t, 0) = x(t, 1) = 30$ for all $t \geq 0$. Our objective is to track a desired damage field α_{des} and probe position p_{des} while ensuring that (a) the probe power is positive and less than $u_{\max} = 1250$, (b) the probe speed is bounded by $v_{\max} = 1$, (c) the probe position remains in $[0.4, 0.6]$, (d) the temperature remains under the maximum temperature $x_{\max} = 80$, and (e) the damage does not exceed the maximum value $\alpha_{\max} = 1.5$.

The output map is given by

$$C := \mathrm{diag}((\varphi_{\xi_i})_{i=1}^{M}, (\varphi_{\xi_i})_{i=1}^{M}, 1). \tag{35}$$



(a) Temperature field under RoLiMPC control.

(b) Degree of damage field under RoLiMPC control.

(c) Probe position and control inputs (probe power and velocity) given by RoLiMPC.

Fig. 1. Simulation results for damage-constrained control of (33) using RoLiMPC. All constraints are honored in the nonlinear system, in spite of linearization and discretization errors.

Since $\boldsymbol{y} = (y_x, y_\alpha, y_p) \in \mathbb{R}^M \times \mathbb{R}_+^M \times \Omega \subset \mathbb{R}^M \times \mathbb{R}^M \times \mathbb{R}$, the output has dimension $2M + 1$. In this example, we take $M = 4$, and we consider a time horizon of $N = 5$. We approximate the resolvent operator by a series expansion with $N_r = 2$, and we take $\delta = 50$.

We now specify the weight operators Q, R, and S for this example. We consider $Q := \operatorname{diag}((Q_x)_{i=1}^M, (Q_\alpha)_{i=1}^M, Q_p)$ for $Q_x = 15$, $Q_\alpha = 50$, $Q_p = 5$, $R = \operatorname{diag}(30, 5)$, and $S = 0$ to limit the number of adjoint evaluations. We have omitted details regarding the adjoint operators and approximation error bounds due to space constraints.

The results are given in Figure 1. Clearly, the output constraints are honored across the entire domain, with the maximum temperature being $77.8 < 80$ in Figure 1a and the maximum degree of damage $1.1 < 1.5$ in Figure 1b. The control inputs satisfy the input constraints, and the probe position stays within $[0.4, 0.6]$ (see Figure 1c). The error approximation bound that we have derived does not exceed 3.1 during the considered time horizon. Note that this approximation error bound was obtained as in Theorem 2 *without* evaluating the full system solution, and its introduction in the RoLiMPC algorithm still yields an adequate solution.

REFERENCES

Dubljevic, S., El-Farra, N.H., Mhaskar, P., and Christofides, P.D. (2006). Predictive control of parabolic PDEs with state and control constraints. *International Journal of Robust and Nonlinear Control*, 16(16), 749–772. doi:10.1002/rnc.1097.

Dubljevic, S. and Humaloja, J.P. (2020). Model predictive control for regular linear systems. *Automatica*, 119, 109066. doi:10.1016/j.automatica.2020.109066.

El-Kebir, H. and Ornik, M. (2021). Online inner approximation of reachable sets of nonlinear systems with diminished control authority. In *Proceedings of the 2021 Conference on Control and Its Applications*, 9–16. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA. doi:10.1137/1.9781611976847.2.

El-Kebir, H., Pirosmanishvili, A., and Ornik, M. (2022a). Online guaranteed reachable set approximation for systems with changed dynamics and control authority. *arXiv:2203.10220 [math.OC]*.

El-Kebir, H., Ran, J., Lee, Y., Chamorro, L.P., Ostoja-Starzewski, M., Berlin, R., and Bentsman, J. (2022b). Minimally invasive live tissue high-fidelity thermophysical modeling using real-time thermography. *IEEE Transactions on Biomedical Engineering*. doi: 10.1109/TBME.2022.3230728. (early access).

Havu, V. and Malinen, J. (2007). The Cayley transform as a time discretization scheme. *Numerical Functional Analysis and Optimization*, 28(7-8), 825–851. doi: 10.1080/01630560701493321.

Humaloja, J.P. and Dubljevic, S. (2018). Linear model predictive control for Schrödinger equation. In *Proceedings of the 2018 Annual American Control Conference*, 2569–2574. IEEE, Milwaukee, WI, USA. doi: 10.23919/ACC.2018.8431686.

Munkres, J.R. (1991). *Analysis on Manifolds*. CRC Press, first edition. doi:10.1201/9780429494147.

Natarajan, V. and Bentsman, J. (2016). Approximate local output regulation for nonlinear distributed parameter systems. *Mathematics of Control, Signals, and Systems*, 28(3), 24. doi:10.1007/s00498-016-0172-7.

Paunonen, L. and Phan, D. (2020). Reduced order controller design for robust output regulation. *IEEE Transactions on Automatic Control*, 65(6), 2480–2493. doi: 10.1109/TAC.2019.2930185.

Toledo, J., Wu, Y., Ramírez, H., and Le Gorrec, Y. (2020). Observer-based boundary control of distributed Port-Hamiltonian systems. *Automatica*, 120, 109130. doi: 10.1016/j.automatica.2020.109130.

Tucsnak, M. and Weiss, G. (2009). *Observation and Control for Operator Semigroups*. Birkhäuser Basel, Basel. doi:10.1007/978-3-7643-8994-9.

Weiss, G. (1994). Regular linear systems with feedback. *Mathematics of Control, Signals, and Systems*, 7(1), 23–57. doi:10.1007/BF01211484.

Wu, Y., Hamroun, B., Le Gorrec, Y., and Maschke, B. (2021). Reduced order LQG control design for infinite dimensional Port-Hamiltonian systems. *IEEE Transactions on Automatic Control*, 66(2), 865–871. doi: 10.1109/TAC.2020.2997373.