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Repulsion-Based p-Dispersion with Distance Constraints in Non-Convex Polygons

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Abstract Motivated by the question of optimal facility placement, the classical p-dispersion problem seeks to place a fixed number of equally sized non-overlapping circles of maximal possible radius into a subset of the plane. While exact solutions to this problem may be found for placement into particular sets, the problem is provably NP-complete for general sets, and existing work is largely restricted to geometrically simple sets. This paper makes two contributions to the theory of p-dispersion. First, we propose a computationally feasible suboptimal approach to the p-dispersion problem for all non-convex polygons. The proposed method, motivated by the mechanics of the p-body problem, considers circle centers as continuously moving objects in the plane and assigns repulsive forces between different circles, as well as circles and polygon boundaries, with magnitudes inversely proportional to the corresponding distances. Additionally, following the motivating application of optimal facility placement, we consider existence of additional hard upper or lower distance bounds on pairs of circle centers, and adapt the proposed method to provide a p-dispersion solution that provably respects such constraints. We validate our proposed method by comparing it with previous exact and approximate methods for p-dispersion. The method quickly produces near-optimal results for a number of containers.

Keywords Dispersion, Packing, Location, Strategic Planning

Declarations

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Code availability: A software implementation of the proposed algorithm is available at `https://github.com/zd6/RApDoP`.

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1 Introduction

The question of optimal placement of a finite number of distinct sites in an area, space, network, polygon, container, or facility presents a motivation for a wide variety of problems in operations research (Soland, 1974; Balachandran & Jain, 1976; Hesse Owen & Daskin, 1998; Dimnaku et al., 2005; Castillo et al., 2008). The primary focus of this paper is the *p-dispersion problem* (Moon & Chaudhry, 1984; Kuby, 1987; Drezner & Erkut, 1995; Dimnaku et al., 2005), which asks to place a *particular number of non-overlapping congruent circles of the maximal radius* within a given set ("container"). In the context of facility placement, such a problem is interpreted as placement of a particular number of sites — each site represented by a point in a Euclidean space — into a given set in such a way that any two sites are at least a certain distance apart.

We preface the subsequent discussion by remarking that prior literature, including that cited in our paper, often refers to p-dispersion as *circle packing*, while also using the same term to denote a related problem of placing a *maximal number of non-overlapping congruent circles of a particular radius*. To avoid confusion, our paper will consistently refer to the problem with a fixed number of circles of a maximum radius as p-dispersion, and the version with a maximal number of circles of a fixed radius as circle packing. Our primary interest is in p-dispersion.

Kuby (1987) was the first to formulate a single mixed-integer program to describe the discrete p-dispersion problem, and use the Big M method to solve the problem. While the language of optimization can be naturally used to similarly pose the general p-dispersion problem, such a problem is provably NP-complete for a general container (Baur & Fekete, 2001). Existing work on this problem has thus focused on analytical solutions for geometrically simple containers (Erkut, 1990; Drezner & Erkut, 1995; Akagi et al., 2018) and methods of obtaining suboptimal solutions (Erkut et al., 1994; Baur & Fekete, 2001; Dimnaku et al., 2005; López & Beasley, 2011; Birgin et al., 2013); Hifi & M'Hallah (2009) provide an extensive study of previous approaches. However — possibly because of computational or geometric difficulties — even the work on suboptimal solutions is largely focused on convex, or at least geometrically simple (e.g., L-shaped), containers. Exceptions are limited in scope and often focus on the related problem of circle packing. Namely,

- Kazakov et al. (2016) provide a heuristic method for p-dispersion based on Lloyd's algorithm that can be applied to non-convex containers, but focus on non-Euclidean metrics and do not perform any actual experiments for non-convex containers,
- Yuan et al. (2018) present some results for p-dispersion non-convex polygons, but their method appears to produce a packing for a *fixed number and fixed radius* of circles, requiring adjustment of the number of circles or circle radii by hand until near-optimal p-dispersion is achieved,
- Baur & Fekete (2001) and Ho (2015) consider p-dispersion and circle packing for non-convex polygons, possibly with circles of differing radii, but their results primarily apply to polygons with sides parallel to the coordinate axes,
- Galiev & Lisafina (2013) consider circle packing for several specific non-convex containers,
- Machchhar & Elber (2017) provide a heuristic method for circle packing in free-form containers with B-spline boundaries.

Compared to the above work, the contribution of our paper is twofold:

(i) we provide a method for p-dispersion that can naturally handle general non-convex polygonal containers and compare it inasmuch as possible with previous work;
(ii) we consider a novel generalization of the p-dispersion problem, motivated by operational needs and regulations in facility placement (Argo & Sandstrom, 2014), where there exist hard bounds on distances between circle centers, and adapt the method in (i) to handle this problem variant.

Our approach to p-dispersion is broadly motivated by two previous approaches. The first approach was used by Machchhar & Elber (2017) for the circle packing problem and Martinez-Rios et al. (2018) for a hybrid of circle packing and p-dispersion in a rectangle, where circle radii are not fixed, but bounded by a priori constraints. Their method relies on a spiral initial placement of circles, followed by "shaking" of the container and subsequent movement of the circle centers, understanding the circles as physical bodies susceptible to gravity. Such logic cannot, however, be directly applied to our problem: the dynamics induced by "shaking" intuitively produce circles that are as tightly packed as possible, corresponding to the objective of the circle packing problem. On the other hand, the p-dispersion problem is intuitively interested in keeping circle centers *as far away as possible*.

The second approach is that of Graham et al. (1998), where near-optimal p-dispersion for a *circle container* is obtained by approximating the problem as one of minimizing the potential energy function between circle centers, in the presence of *repulsive forces* between the circles. However, such an approximated problem is still solved by usual optimization methods, limiting its applicability to simple, convex containers, and not allowing for the existence of additional constraints.

Combining the above two approaches, our work proposes a method of p-dispersion relying on moving circles for *general polygonal containers*, with dynamics based on repulsive forces between the circles. As in Graham et al. (1998), we assign repulsive forces with distance-depending magnitudes: for instance, in order to enable circles to grow, circles that are close to each other will be enticed by their corresponding repulsive force to move away from each other. Unlike Graham et al. (1998), we also consider repulsive forces between circles and the container boundary.

In addition to providing a novel method of heuristically solving the p-dispersion problem for free-form non-convex polygons, we consider an additional problem feature not investigated in previous literature. Namely, motivated by the application of site placement in an area, space, or facility, there may exist hard bounds on distances between particular circle centers, motivated by potential operational needs and regulations, such as clearance distances for safe chemical storage in a facility (Argo & Sandstrom, 2014), minimum separation distances for packages containing radioactive materials in passenger-carrying aircraft (49 C.F.R. §175.701, 2021), or maximum separation distances between portable fire extinguishers in an area for fire prevention (29 C.F.R. §1910.157, 2021). We tackle these additional constraints by determining initial circle center positions that satisfy the constraints, and restricting the motion generated by the dynamics model to ensure that the constraints remain satisfied.

The outline of the paper is as follows. In Section 2 we provide the motivation and formal statements of the problems that this paper considers. Section 3 provides the main contribution of the paper: a formal description of the repulsion-based approach to maximal dispersion. In addition to introducing pairwise repulsive forces between circles, repulsive forces between circles and container boundary are introduced in Section 3.1, and Section 3.2 discusses satisfaction of hard bounds on pairwise distances between circle centers. The numerical implementation of our overall approach is described and validated in Section 4, where we present our solutions to the p-dispersion problem for a variety of convex and non-convex polygonal sets, and compare them with known p-dispersion results.

Notation. Symbol $\mathbb{N}$ denotes the set of positive integers. For $n \in \mathbb{N}$, we denote set $\{1, \ldots, n\}$ by $[n]$. We denote the usual Euclidean distance on $\mathbb{R}^2$ by $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, and by a standard abuse of notation, $d(x, A)$ denotes the distance between point $x \in \mathbb{R}^2$ and $A \subseteq \mathbb{R}^2$, i.e., $d(x, A) = \inf\{d(x, y) \mid y \in A\}$. Naturally, we let $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}$ denote the Euclidean norm. The line segment between points $x, y \in \mathbb{R}^2$ is denoted by $\overline{xy}$. The diameter of a set $A \subseteq \mathbb{R}^2$ is given by $\operatorname{diam}(A) = \sup\{d(x, y) \mid x, y \in A\}$. The boundary of the set A is denoted by ∂A. The complement $\mathbb{R}^2 \backslash A$ of a set $A \subseteq \mathbb{R}^2$ is denoted by A^c. A closed ball in $\mathbb{R}^2$ with the center at $x \in \mathbb{R}^2$ and radius r is denoted by $\mathbb{B}(x, r)$.

2 Motivation and Problem Statement

The problems considered in this paper are variants of the classical *continuous p-dispersion* problem (Drezner & Erkut, 1995; Dimnaku et al., 2005). Such a problem asks to find p points (often called *sites*) within a compact set (i.e., *container*) $A \subseteq \mathbb{R}^n$ such that the minimal distance between any two of those points is maximized. In our case, considered by Baur & Fekete (2001), we also require that the distance of the sites from the container boundary be maximized. Namely, we require each site to be surrounded by an open ball of radius r that is *entirely contained within the container* and not intersecting any of the balls surrounding other sites; such an interpretation is motivated by purposes of defense and identification of intruder and visitor intentions (Joint Chiefs of Staff, 1996). The above problem is formalized as follows.

Problem 1 (p-dispersion with boundary) Let $p \in \mathbb{N}$ and $A \subseteq \mathbb{R}^2$ compact. Find the maximal $r \geq 0$ for which there exist $s_1, \ldots, s_p \in A$ such that the following properties hold:

(i) $d(s_i, s_j) \geq 2r$ for all $i, j \in [p]$, $i \neq j$.
(ii) $d(s_i, A^c) \geq r$ for all $i \in [p]$.

As mentioned, the p-dispersion problem occasionally considers satisfying only property (i). The methods that we will use to work on the above problem can directly be applied to such a variant.

As noted by Erkut & Neuman (1989), the problem of optimal site placement often comes with additional constraints on the choice of sites. Namely, for operational reasons, the sites may be required to be placed at a distance no greater than some maximal bound, and for safety reasons, some sites may be required to be separated by at least some predetermined distance (Argo & Sandstrom, 2014). We will thus additionally consider a further requirement:

(iii) $d^l_{ij} \leq d(s_i, s_j) \leq d^u_{ij}$ for all $i, j \in [p]$, $i \neq j$,

where $d^l_{ij}, d^u_{ij} \in [0, +\infty]$. When considering properties (i), (ii), and (iii) together, we will call the corresponding problem *p-dispersion with boundary and hard bounds.*

The p-dispersion problem, in our case possibly with boundary and hard bounds, can be posed (Drezner & Erkut, 1995) as an optimization problem:

$$\begin{aligned}
\min \quad & -r && \text{(1a)}\\
\text{such that} \quad & s_1, \ldots, s_p \in A, && \text{(1b)}\\
& 2r - \|s_i - s_j\| \leq 0 \text{ for all } i, j \in [p],\ i \neq j, && \text{(1c)}\\
& r - d(s_i, A^c) \leq 0 \text{ for all } i \in [p], && \text{(1d)}\\
& \|s_i - s_j\| - d^u_{ij} \leq 0 \text{ for all } i, j \in [p],\ i \neq j, && \text{(1e)}\\
& d^l_{ij} - \|s_i - s_j\| \leq 0 \text{ for all } i, j \in [p],\ i \neq j, && \text{(1f)}
\end{aligned}$$

where the decision variables are $r \in \mathbb{R}$ and $s_1, \ldots, s_p \in \mathbb{R}^2$. Constraint (1c) corresponds to requirement (i) above, constraint (1d) to requirement (ii), and constraints (1e) and (1f) to requirement (iii).

Noting that function $d(\cdot, A^c)$ is continuous regardless of set A (Ciarlet, 2013) and that requirements (1b) and (1c) imply $r \leq \operatorname{diam}(A)$, the solution of (1) indeed exists, as a maximum of a continuous function on a closed and bounded set. Nonetheless, the feasible set in problem (1) is non-convex. While there has been some success at providing optimal solutions to the p-dispersion problem in the case of extremely simple containers A, the p-dispersion problem is thus generally computationally infeasible to solve exactly (Baur & Fekete, 2001; Dimnaku et al., 2005), yielding a natural desire to obtain computationally feasible near-optimal solutions. We will introduce a method that avoids considering the optimization problem (1), instead aiming to find an approximate solution by solving an ordinary differential equation.

We proceed by describing our proposed approach to the p-dispersion problem with boundary. In Section 3.2, we will generalize this approach by allowing for the existence of hard bounds, which are modeled by placing restrictions on motions induced by the repulsive forces between sites.

3 Repulsion-Based Dispersion

The fundamental idea of our work, motivated by blending the approaches of Machchhar & Elber (2017) and Graham et al. (1998), is to model the circles $\mathbb{B}(s_i, r)$, $i \in [p]$, as physical bodies that are subject to move over time under the defined repulsive forces between different circles and between circles and set boundaries. Namely, each circle center satisfies the ordinary differential equation (ODE)

$$\ddot{s}_i(t) = f_A^i(s_1(t), \dots, s_p(t)), \tag{2}$$

where f_A^i encodes the total repulsive force acting on $\mathbb{B}(s_i, r)$ from other circles and boundary of the container A. The radius $r(t)$ is then computed as the maximal possible radius that ensures the satisfaction of conditions (i) and (ii) in the p-dispersion with boundary problem:

$$r(t) = \min \left\{ \min_{i \in [p]} d(s_i(t), A^c), \min_{i,j \in [p], i \neq j} d(s_i(t), s_j(t))/2 \right\}. \tag{3}$$

The intuition behind designing repulsive forces f_A^i is illustrated in Fig. 1. Namely, total repulsive force $f_A^i(s_1, \dots, s_p)$ is given by

$$f_A^i(s_1, \dots, s_p) = \sum_{j \neq i} F(s_j, s_i, r) + F_{\partial A}(s_i, r), \tag{4}$$

where $F(s_j, s_i, r)$ is the repulsive force applied by circle $\mathbb{B}(s_j, r)$ on the circle $\mathbb{B}(s_i, r)$, and $F_{\partial A}(s_i, r)$ is the total repulsive force applied by the sides in the container boundary ∂A on the circle $\mathbb{B}(s_i, r)$. Forces naturally act in the direction away from their source, and their magnitude increases as the destination circle $\mathbb{B}(s_i, r)$ becomes closer to the source. From our intuition of repulsive forces keeping circles as far away from each other and from ∂A as much as possible, we expect that a good suboptimal solution to Problem 1 may then be obtained as $s_1(T), \dots, s_p(T)$ and $r(T)$ for some large T, with any initial values $s_1(0), \dots, s_p(0) \in A$ and any $\dot{s}_1(0), \dots, \dot{s}_p(0)$.

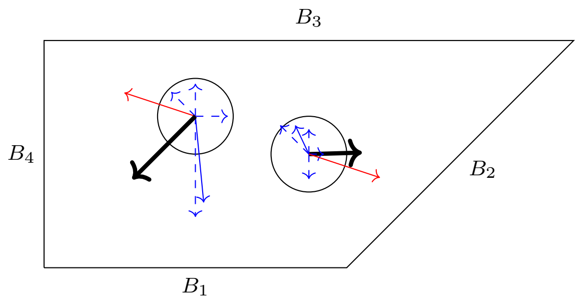


Fig. 1 Illustration of the repulsive forces acting on two circles in a polygon A. The force generated by each side B_i of A is denoted with a dashed blue line. Their sum, i.e., the total force generated by ∂A is denoted by a solid blue line. The repulsive forces between the circles are denoted by a red line. The total force acting on each circle is denoted by a thick black line. As the circle on the left is closest to B_3 and to the other circle, the dominant repulsive forces acting on it are coming from those two sources. The dominant force acting on the right circle is the repulsive force between two circles. For visibility reasons, some short blue lines are not proportional to force magnitude.

Let us first formalize the repulsive force between two circles. Force $F(s_j, s_i, r)$ is given by

$$F(s_j, s_i, r) = \frac{\mu_C}{(d(s_i, s_j) - 2r)^2} \frac{s_i - s_j}{d(s_i, s_j)}, \tag{5}$$

where $\mu_C > 0$. This definition is naturally motivated by our previous discussion: $F(s_j, s_i, r)$ pushes circle $\mathbb{B}(s_i, r)$ away from $\mathbb{B}(s_j, r)$, with the magnitude $\mu_C/(d(s_i, s_j) - 2r)^2$. In other words, as $\mathbb{B}(s_i, r)$ and $\mathbb{B}(s_j, r)$ get closer to touching, the magnitude of the repulsive force increases to $+\infty$. The value of μ_C indicates the *repulsive strength* of the circles; naturally, different values of μ_C yield slightly different configurations of s_i and r.

The exact definition in (5) is not the only option corresponding to our motivation; nonetheless, the magnitude of the repulsive force increasing to $+\infty$ as circles get closer ensures that circles will never touch, and their radii, given by (3), will never decrease. We note that, if r as defined in (3) exactly equals some $d(s_i, s_j)/2$, the magnitude of the repulsive force from (5) will be $+\infty$; to avoid such an occurrence, in practice we restrict F to not exceed a certain maximal magnitude. Having defined the first part of the total repulsive force f_A^i from (4), let us now consider the repulsive forces generated by the boundary ∂A.

3.1 Boundary Repulsion

Throughout this section, we assume A to be a simply connected, possibly non-convex polygon, i.e., a sequence of vertices $(v_1, \ldots, v_{m-1}, v_m, v_1)$ with sides $B_j = \overline{v_j v_{j+1}}$ for $i \in [m-1]$ and $B_m = \overline{v_m v_1}$. We naturally assume that sides do not intersect, except $B_j \cap B_{j+1} = \{v_{j+1}\}$ for $j \in [m-1]$ and $B_m \cap B_1 = \{v_1\}$. For $s_i \in A$, let $b_j i$ denote the point on the side B_j that is closest to s_i by b_{ji}. We emphasize that b_{ji} needs to lie on the side itself, and not just on the line containing B_j. Hence, if there is no perpendicular from s_i to B_j, it will equal one of the two endpoints of B_j.

We will define the repulsive $F_{\partial A}(s_i, r)$ of ∂A acting on $\mathbb{B}(s_i, r)$ as the sum of repulsive forces $F_{B_j}(s_i, r)$ generated by sides B_j. The repulsive force from side B_j naturally acts in the direction perpendicular to B_j and pointing inside ∂A. If there exists a perpendicular to B_j going from some point in B_j to s_i, the origin point of such a perpendicular is exactly b_{ji}. Analogously to the repulsive force (5), it may thus seem natural to define

$$F_{B_j}(s_i, r) = \frac{\mu_B}{(d(s_i, b_{ji}) - r)^2} \frac{s_i - b_{ji}}{d(s_i, b_{ji})}, \tag{6}$$

where $\mu_B > 0$.

Upon closer inspection, definition (6) might not, however, always be intuitive. Fig. 2 provides an illustration of the issues; for instance, there is no simple physical motivation for assigning any particular repulsive force from B_5 or B_6 to s_i.

We consequently make a small adjustment for non-convex polygons. We define

$$F_{\partial A}(s_i, r) = \sum_{j=1}^{m} F_{B_j}(s_i, r), \tag{7}$$

and distinguish between three cases:

(a) *There exists a perpendicular to B_j going from b_{ji} to s_i.*

We consider two halfspaces generated by the line containing B_j. Namely, if we traverse ∂A in the counterclockwise manner, points adjacent to B_j in the halfspace to the left of B_j will be contained in A, while points adjacent to B_j in the right halfspace of B_j will be in A^c (LaValle, 2006). We consequently have the intuition that, if s_i is in the left halfspace of B_j, it makes sense to try to repulse s_i from B_j. If s_i is in the right halfspace of B_j, the notion of a repulsive force does not make much intuitive sense. Accordingly, we define

(a1) $F_{B_j}(s_i, r)$ as in (6) if s_i is in the open left halfspace with respect to B_j,

(a2) $F_{B_j}(s_i, r) = 0$ otherwise.

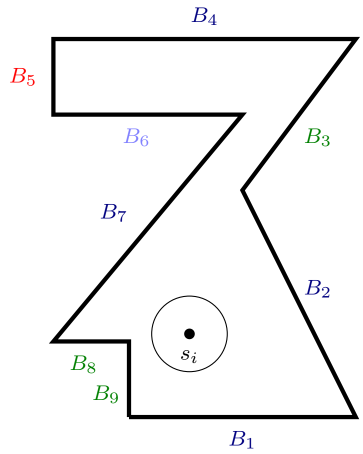


Fig. 2 A nonconvex polygon A with sides $B_1, \ldots, B_9$, and a circle $\mathbb{B}(s_i, r)$ within it. Blue letters denote sides that illustrate case (a) below, with sides that illustrate (a1) denoted in dark blue, and the side that illustrates (a2) denoted in light blue. Green letters denote sides that illustrate case (b) below. Red letter denotes the side that illustrates case (c) below.

Sides B_1, B_2, B_4, and B_7 in Fig. 2 illustrate case (a1). Side B_6 illustrates (a2).

(b) *There does not exist a perpendicular to B_j going from b_j to s_i, but there exists a* line of sight *between b_{ji} and s_i, i.e., the segment $\overline{b_{ji}s_i}$ lies entirely in A.*

We define $F_{B_j}(s_i, r)$ as in (6).

Sides B_3, B_8, and B_9 in Fig. 2 illustrate case (b).

(c) *There does not exist a perpendicular to B_j going from b_{ji} to s_i, and there does not exist a line of sight between s_i and b_{ji}.*

In this case, we define $F_{B_j}(s_i, r) = 0$.

Side B_5 in Fig. 2 illustrates case (c).

We note that, because cases (a2) and (c) are trivially not possible in a convex container, $F_{B_j}(s_i, r)$ is always defined by (6) for convex polygons. As with the repulsion force between circles, there may be multiple ways of defining the forces in this section, even following the same intuition. We will nonetheless show that our definition produces results remarkably close to optimal in many previously considered cases, and meaningful results for non-convex polygons that have not been considered previously. We now continue with the last element of our solution: possible existence of hard distance bounds.

3.2 Distance Bounds

We now return to the novel element of the p-dispersion problem introduced in Section 2. Namely, motivated by applications in facility location (Argo & Sandstrom, 2014), requirement (iii) in that section states that there may exist d^l_{ij} and/or d^u_{ij} such that circle centers are required to satisfy $d(s_i, s_j) \geq d^l_{ij}$ and $d(s_i, s_j) \leq d^u_{ij}$.

Naturally, solving (2) does not immediately guarantee satisfaction of the above constraints. We now present a theoretically sound and physically meaningful, if computationally expensive, way of ensuring that the constraints are satisfied, and a computationally cheaper interpretation of the same idea. We imagine circle centers connected by *length-constrained links*, with large enough forces at the end of the joint disabling the circle centers from breaking the constraints. We illustrate the definition of the length-constrained link in Fig. 3.

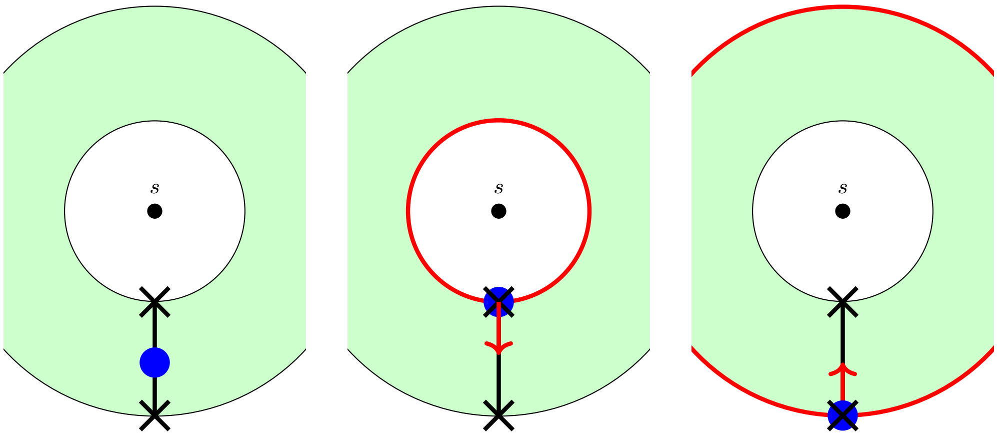


Fig. 3 A length-constrained link approach to hard constraints on circle distances. The center of the blue circle is constrained to stay within a particular distance of point s. In the left picture, neither the upper nor the lower bound on the distance are achieved, so there are no forces related to this constraint. In the other two pictures, the distance bounds are achieved, so a force acts on the blue circle to prevent it from leaving the constrained distance range.

Given that constraints may couple several circles together to form a network of length-constrained links, computing the circle center movements may be computationally expensive and numerically challenging. We thus propose a computationally simpler interpretation of the above physical idea.

As we discuss in Section 4, in our implementation we are solving (2) by discretizing it into a difference equation. Our solution to continued satisfaction of constraints is thus to validate each possible move with respect to defined constraints. Namely, at each time step, we sort the circles with respect to the magnitude of total repulsion force applied to them (not considering any forces related to the hard bounds). We then verify whether the constraints will remain satisfied if the circle with the highest total repulsion force moves according to its repulsion force. If they will remain satisfied, the circle is allowed to make its move. If they would not remain satisfied, the circle remains in place. We proceed onwards with the circle with the second total repulsion force, and continue onwards. While the proposed method, thus, potentially occasionally "overrides" the repulsion forces defined in (4), it obviously satisfies the following claim.

Observation 1 *Let* $s_1(0), \ldots, s_p(0) \in A$, *with* $d^l_{ij} \leq d(s_i(0), s_j(0)) \leq d^u_{ij}$ *for all* $i, j \in [p]$. *Then* $d^l_{ij} \leq d(s_i(t), s_j(t)) \leq d^u_{ij}$ *for all* $t \geq 0$ *and all* $i, j \in [p]$.

This discussion concludes our description of the proposed dispersion method; we proceed to validate the method on a number of classical and novel examples.

4 Numerical Implementation

Ordinary differential equation (2), with the right hand side defined in (3)–(7), is difficult to solve analytically. Additionally, standard ODE solvers may not produce trustworthy results in computationally feasible time due to *stiffness* of (2): the right hand side is not bounded in magnitude from above and can also be arbitrarily small. We thus present a modification of the method given in Section 3 amenable to numerical computations. As a consequence of Observation 1, such a modification, unlike naive methods relying on ODE solvers, ensures that if the circle centers are initially located within A and satisfy hard bounds on their distance, they will continue to do so throughout the simulation run.

The implementation adapts the proposed method in the following ways:

- Instead of the second-order ODE (2), we consider the *first-order* difference equation

$$s_i(t + \Delta t) = s_i(t) + \Delta t f_A^i(s_1(t), \ldots, s_p(t)).$$

  The repulsive forces thus act directly on the circle velocity, and not acceleration. Because of the switch to discrete time, we need to manually ensure that the circles do not leave polygon A; step size Δt is consequently decreased as needed.
- The magnitude of repulsive forces f_A^i defined in (4) and (7) is bounded from above, to avoid forces with infinite magnitude in the case where two circles or a circle and ∂A are touching.
- For computational reasons, two circles or a circle and a side that are sufficiently far away or do not have a line of sight are assumed to yield no repulsive force.

The described implementation, with a user interface allowing for simple use, is available at `https://github.com/zd6/RApDoP`.

We now proceed to discuss the results obtained by our method. As mentioned, the problem of p-dispersion for convex containers has been thoroughly investigated. For instance, in the case of a unit square container, optimal dispersions of p equal circles for $p \leq 20$ have been found (Friedman, 2019). Table 1 compares the provably maximal radius r^* with results of our repulsion-based method. In our method, we simulated the circle motion from 2000 sets of initial points with varying μ_B and μ_C, and with no more than 2000 time steps. Our method yields results that are very close to the provably optimal solutions, with minute difference that could possibly be attributed to numerical details of our implementation.

We next report our results for p-dispersion in an L-shaped container — to the best of our knowledge, the only class of non-convex containers thoroughly considered prior to our work. We compare our solutions, for $p \in \{3, \ldots, 16\}$, to the best known solutions (Friedman, 2019). The results are given in Table 2. Our method again produces near-optimal solutions.

Turning to other papers for comparisons, we note that while the previously proposed method of Kazakov et al. (2016) can be applied to p-dispersion in general non-convex containers, their paper does not contain any numerical examples for such containers. Furthermore, it is impossible to produce such experiments using solely the information in the paper, because the paper's implementation lacks the discussion of the choice of design parameters. Thus, in Table 3 we compare our results to those of Yuan et al. (2018), where a number of complicated convex and non-convex containers are considered. Given that the method of Yuan et al. (2018) seems to involve tweaking the circle radius by hand, our method easily exceeds the results proposed by Yuan et al. (2018) in all but one case, where the two methods are equal. Our method thus produces the best known results for the considered containers.

In addition to producing effectively optimal solutions in the above cases, by avoiding a nonlinear optimization problem — a computationally expensive approach present in much of the previous work, including Graham et al. (1998) — our method is able to produce results for a large p and general non-convex polygonal containers with a large number of sides. For instance, obtaining a dispersion of 16 circles inside an L-shaped container, for a single set of initial points, takes around 0.42 seconds on a commercial Intel i7-9750H processor running MATLAB R2019b.

p	Optimal radius	Our method	**Percent difference**	Mean	Median	SD	Range
1	0.50000	0.49996	**0.00800**	0.49986	0.49984	0.00005	0.00015
2	0.29289	0.29289	**0.00000**	0.29289	0.29289	0.00001	0.00004
3	0.25433	0.25432	**0.00393**	0.25430	0.25430	0.00001	0.00004
4	0.25000	0.25000	**0.00000**	0.25000	0.25000	0.00000	0.00000
5	0.20711	0.20710	**0.00483**	0.20709	0.20709	0.00001	0.00003
6	0.18768	0.18762	**0.03197**	0.18761	0.18761	0.00001	0.00004
7	0.17446	0.17441	**0.02866**	0.17439	0.17439	0.00001	0.00003
8	0.17054	0.17053	**0.00586**	0.16877	0.16854	0.00063	0.00278
9	0.16667	0.16667	**0.00000**	0.16666	0.16666	0.00000	0.00002
10	0.14820	0.14813	**0.04723**	0.14699	0.14684	0.00081	0.00277
11	0.14240	0.14224	**0.11236**	0.14090	0.14041	0.00075	0.00247
12	0.13996	0.13991	**0.03572**	0.13846	0.13987	0.00192	0.00549
13	0.13399	0.13394	**0.03732**	0.13205	0.13192	0.00155	0.00410
14	0.12933	0.12928	**0.03866**	0.12789	0.12802	0.00101	0.00316
15	0.12717	0.12641	**0.59763**	0.12549	0.12550	0.00046	0.00143
16	0.12500	0.12499	**0.00800**	0.12132	0.11967	0.00323	0.00811
17	0.11720	0.11698	**0.18771**	0.11569	0.11569	0.00118	0.00343
18	0.11552	0.11517	**0.30298**	0.11269	0.11270	0.00100	0.00382
19	0.11227	0.11210	**0.15142**	0.11050	0.11079	0.00111	0.00343
20	0.11138	0.11132	**0.05387**	0.10895	0.10905	0.00158	0.00497

Table 1 Comparison of optimal solutions for p-dispersion in a unit square (Friedman, 2019) with the proposed method. The result of our method is obtained as the best of the outcomes for 2000 sets of initial points. Statistical characteristics — mean, median, standard deviation, and range — of the 2000 experiments are provided in the last four columns of the table.

p	Best known radius	Our method	**Percent difference**	Mean	Median	SD	Range
3	0.50000	0.49999	**0.00200**	0.49998	0.49998	0.00001	0.00003
4	0.37710	0.37687	**0.06099**	0.37679	0.37680	0.00005	0.00018
5	0.34600	0.34563	**0.10694**	0.34545	0.34541	0.00010	0.00030
6	0.31170	0.31152	**0.05775**	0.30981	0.31143	0.00241	0.00620
7	0.29470	0.29446	**0.08144**	0.29319	0.29277	0.00071	0.00198
8	0.28110	0.28084	**0.09249**	0.27855	0.27722	0.00218	0.00732
9	0.27290	0.27273	**0.06229**	0.26907	0.26865	0.00279	0.00875
10	0.26220	0.26200	**0.07628**	0.25607	0.25479	0.00330	0.00920
11	0.25440	0.25017	**1.66274**	0.24999	0.24998	0.00006	0.00026
12	0.25000	0.24998	**0.00800**	0.24996	0.24996	0.00001	0.00004
13	0.22700	0.22667	**0.14537**	0.22319	0.22350	0.00250	0.00864
14	0.22010	0.22001	**0.04089**	0.21406	0.21406	0.00358	0.01276
15	0.21250	0.21153	**0.45647**	0.20679	0.20712	0.00281	0.01156
16	0.20760	0.20535	**1.08382**	0.19927	0.20036	0.00384	0.01334

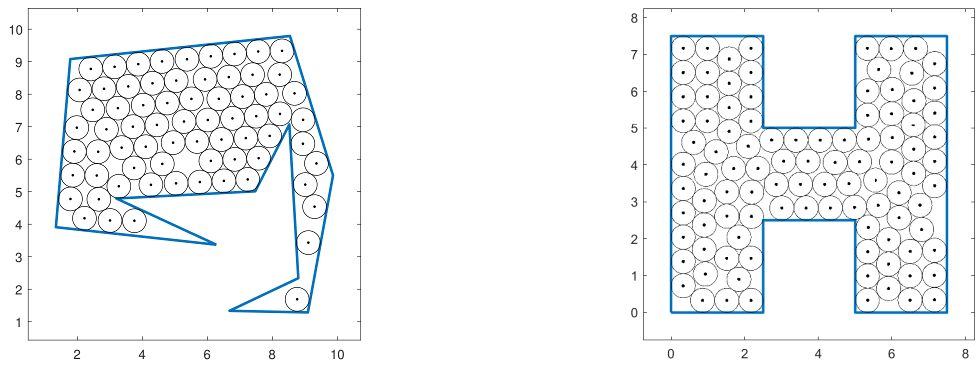
Table 2 Comparison of best known solutions for p-dispersion in an L-shaped container (Friedman, 2019) with the proposed method. The result of our method is obtained as the best of the outcomes for 2000 sets of initial points. Statistical characteristics of the 2000 experiments are provided in the last four columns of the table.

Fig. 4 illustrates these abilities of our method. Namely, Fig. 4a features dispersal of 70 circles in a non-convex container, while Fig. 4b gives a dispersal of 100 circles in an H-shape container. While it is impossible to compare our results to previous work, which never considered such a large number of circles or complex containers, the circles indeed seem to fill the containers near-optimally.

Fig. 5 considers the same shapes and numbers of circles as in Fig. 4, but also introduces hard bounds for the two containers. Namely, we require $d^l_{12} = 0.1$ and $d^u_{12} = 0.6$ for the left container and $d^l_{12} = 0.2$, $d^u_{12} = 0.6$, $d^l_{23} = 0.65$, $d^u_{23} = 10.0$ for the right. For the both containers, the upper bound $d^u_{12} = 0.6$ implies that the maximum value of $2r$ is limited to 0.6, therefore, the circle radii cannot be greater than 0.3, even though there is still empty space in the container. As Fig. 5 shows, our algorithm indeed produces a

Vertices	Convexity	p	Radius of Yuan et al. (2018)	Radius from our method
{(0,0), (19,0), (10,19)}	Convex	43	1	1.0253
{(0,0), (23,0), (23,13), (5,14)}	Convex	29	1.5	1.5195
{(0,0), (26,0), (12,8), (8,16)}	Non-convex	18	1.5	1.5
{(0,0), (14,0), (18,5), (11,19), (-10,13)}	Convex	20	2	2.0525
{(0,0), (18,0), (20,18), (8,13), (-1,19)}	Non-convex	17	2	2.1107

Table 3 Comparison of p-dispersion results for containers and numbers of circles considered by Yuan et al. (2018) with the proposed method.



(a) Dispersal of 70 circles in a non-convex container with 11 sides. The resulting radius equals 0.36610.

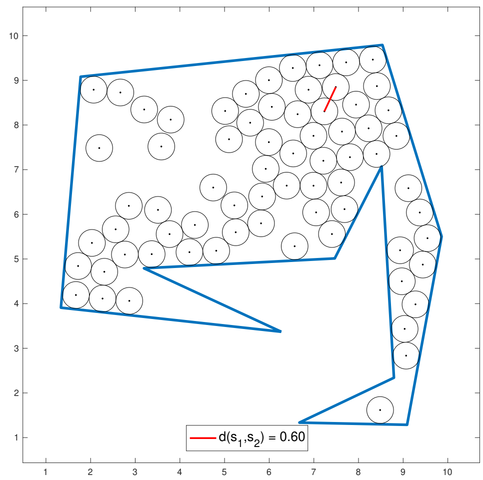
(b) Dispersal of 100 circles in a non-convex H-shape container. The resulting radius equals 0.32867.

Fig. 4 Dispersal of circles in non-convex containers.

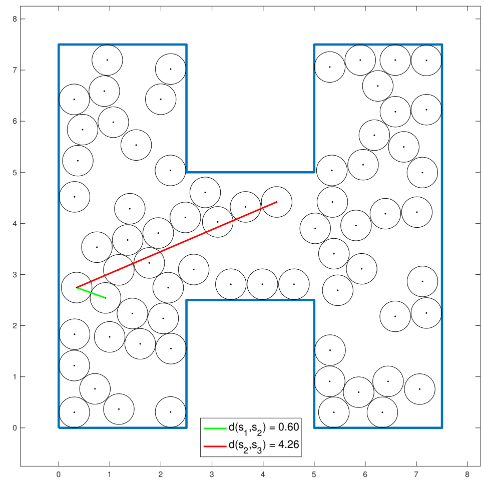
provably optimal p-dispersion given these constraints. We emphasize that, in the current setup, a bound on the distance between any pair of circles affects *all* circles, because all radii are the same. A variant of the problem that we briefly discuss at the end of our paper would allow for circles to have differing radii, with distance constraints affecting only some of them.

Now, to return to the motivating application of optimal facility placement, Fig. 6 displays three examples of dispersal of large number of circles into exceedingly complicated containers with 12–28 sides, motivated by real-world facilities.The leftmost container in Fig. 6 is derived from the boundaries of the Northern Plains Commerce Center industrial park (Bismarck Mandan Chamber EDC, 2009) located in Bismarck, North Dakota. The middle container is inspired by the floorplan of the CSA Fieldhouse in Prince Frederick, Maryland (Calvert Soccer Association, 2017). Finally, the shape of the rightmost container resembles the layout of Langdon High School in Langdon, North Dakota (Scott Strasser, 2020). In all of the cases, our method produces dispersals that appear to be near-optimal.

Finally, we provide a brief discussion of the computational time performance of our method. While hard results are difficult to obtain because of the iterative nature of the method, Fig. 7 illustrates the time to compute dispersal of n circles in a unit square, 20 circles in a regular convex polygon with n sides, and 20 circles in a star-shaped non-convex polygon with n sides. The proposed fitting functions — quadratic, linear, and exponential — were chosen without theoretical grounding but solely on the observation of which functions seem to portray the data well. We make no claim that the true complexity of the algorithms is quadratic, linear, or exponential. The tests were run on a personal computer with a Ryzen 3800 GPU with



(a) Dispersal of 70 circles in a non-convex container with 11 sides. The distance constraints are: $d^l_{12} = 0.1$ and $d^u_{12} = 0.6$. The resulting radius equals 0.3000.



(b) Dispersal of 70 circles in an H-shaped container. The distance constraints are: $d^l_{12} = 0.2$, $d^u_{12} = 0.6$, $d^l_{23} = 0.65$, and $d^u_{23} = 10.0$. The resulting radius equals 0.3000.

Fig. 5 Dispersal of circles in non-convex containers, with hard bounds on distances.

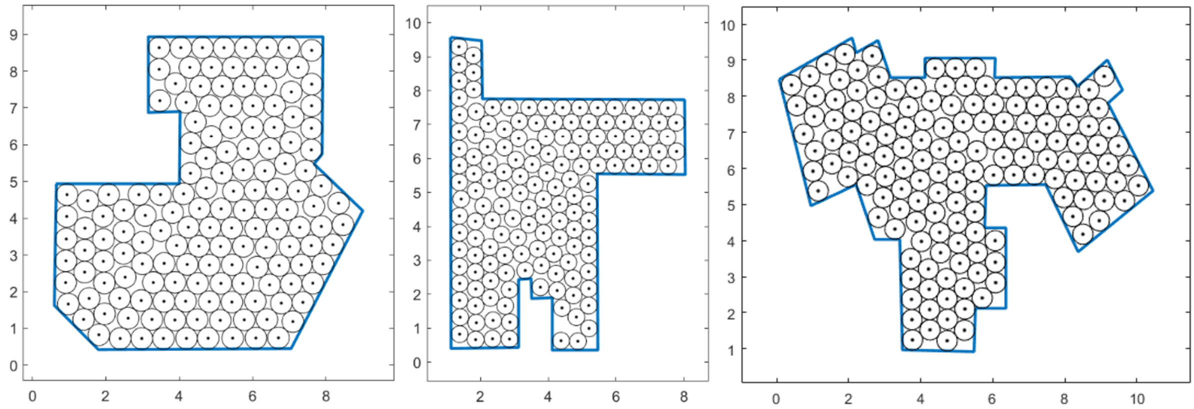
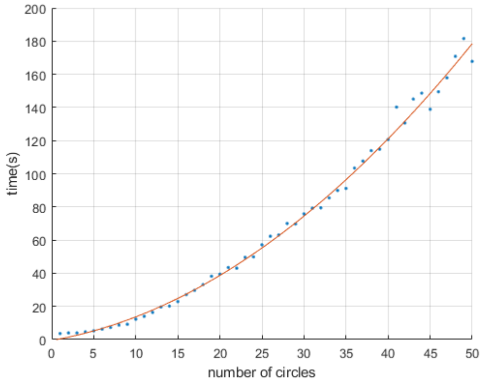
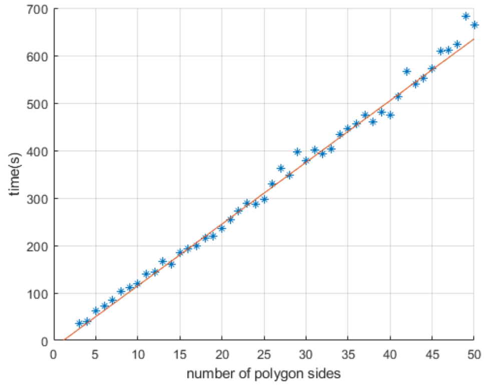


Fig. 6 Dispersal of 150 circles in non-convex containers with 12, 14, and 28 sides, from left to right. The resulting radii equal 0.2941, 0.2487, and 0.2801, respectively. In all implementations, we used 10 sets of initial points, 15000 time steps, and varying parameters μ_B and μ_C.

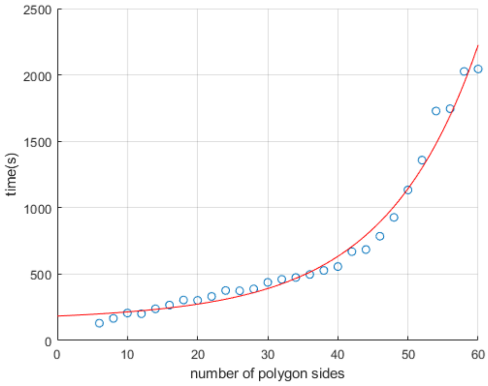
12 threads at 4GHz on MATLAB 2021b. For all three cases, we simulated the circle motions from 500 sets of initial points.



(a)



(b)



(c)

Fig. 7 Test of computation performance. (a) Computation time for dispersing n circles in a unit square for $n \leq 50$. The performance can be best fitted with a quadratic polynomial $y = .054x^2 + .88x - .51$ shown in red. (b) Computation time for dispersing 20 circles in a n-sided regular convex polygon for $n \leq 50$. The performance can be fitted with a linear equation $y = 13x - 15$ shown in red. (c) Computation time for dispersing 20 circles in a n-sided non-convex polygon in the shape of an $\frac{n}{2}$-pointed star, for $n \leq 60$. The performance can be best fitted with a two-term exponential $y = 163.1e^{.004449x} + 20.84e^{0.07617x}$ shown in red.

5 Conclusions and Future Work

This paper studies the classical p-dispersion problem. It proposes a method that finds a suboptimal solution to the p-dispersion problem for free-form non-convex polygonal containers. The proposed method, motivated primarily by the dynamics of a p-body problem, treats circle centers as continuously moving objects in the plane driven by repulsive forces defined between different circles, as well as circles and boundary sites. Such an approach is also readily adaptable to deal with a novel generalization of the p-dispersion problem motivated by the application of optimal facility placement, where there exist additional hard upper and lower distance bounds on pairs of circle centers; the adaptation considers circle centers as objects connected by a length-constrained link. The software implementation of our method, made available online and with a simple user interface, is capable of handling a large number of circles in complex containers, both with and without additional hard bounds, and produces near-optimal solutions for previously considered p-dispersion problems.

We see several potential avenues for future improvement of the presented work. First of all, a better initialization algorithm can be developed to generate initial points more effectively, perhaps motivated by the spiral pattern employed by Machchhar & Elber (2017). Such an improvement to initialization is especially important in the presence of hard distance bounds, when our method relies on circle centers initially satisfying the bounds. Additionally, the current implementation uses an ad hoc algorithm to discretize the underlying ODE; a preferred course of action would be to implement or use a more general sophisticated ODE solver capable of adequately dealing with the problem stiffness and computational complexity. Similarly, a theoretical discussion of the optimal choice of parameters μ_B and μ_C, instead of choices made on the basis of intuition and observation, would likely yield near-optimal results with a comparably smaller computational effort. For both practical and academic reasons, it would also be meaningful to extend the proposed method and its implementation to allow for non-polygonal shapes, such as containers with holes or free-form containers with a curved boundary, possibly described by B-splines. The effort on containers with holes can then be simply used to handle the problem of previously existing facilities, where new facilities need to be added in a manner that disperses them from each other and from already existing facilities; existing facilities can just be encoded as holes in a container. Future endeavors along B-splines would be important for the approximation of regions without straight lines, such as political entities or geographical land forms. Our current method would rely on adding additional sides to a polygon to better approximate such regions, which increases the computational expense.

Finally, for real-world facility location applications, it is worth considering some variations of our problem. We note two possible extensions: the first one, mentioned in a previous section, asks to maximize the radius for *most* circles except those constrained by the hard distance bounds, or, more generally, maximize some function of different radii. Such a problem naturally arises in scenarios where facilities are not operationally entirely equal, i.e., may have different needs and capabilities. The high-level idea of our method, using repulsive forces to ensure dispersion, can be readily adapted to such a problem, but repulsive forces would need to be modified to respond to differing constraints in different facilities. Finally, it is possible to adapt the presented method into a multi-objective problem where dispersion is only a secondary objective (Maliszewski et al., 2012) by assigning weighted repulsive strength for each objective and looking for a new optimum. Again, while the high-level method remains the same, the problem can become computationally demanding as more trade-offs are considered, and a proper choice of weights remains to be discussed.

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