Quantitative Resilience of Linear Driftless Systems*

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Abstract

This paper introduces the notion of quantitative resilience of a control system. Following prior work, we study linear driftless systems enduring a loss of control authority over some of their actuators. Such a malfunction results in actuators producing possibly undesirable inputs over which the controller has real-time readings but no control. By definition, a system is resilient if it can still reach a target after a partial loss of control authority. However, after a malfunction, a resilient system might be significantly slower to reach a target compared to its initial capabilities. We quantify this loss of performance through the new concept of quantitative resilience. We define such a metric as the maximal ratio of the minimal times required to reach any target for the initial and malfunctioning systems. Naïve computation of quantitative resilience directly from the definition is a complex task as it requires solving four nested, possibly nonlinear, optimization problems. The main technical contribution of this work is to provide an efficient method to compute quantitative resilience. Relying on control theory and on two novel geometric results we reduce the computation of quantitative resilience to a single linear optimization problem. We demonstrate our method on an opinion dynamics scenario.

1 Introduction

When failure is not an option, critical systems are built with sufficient actuator redundancy [20] and with fault-tolerant controllers [24]. These systems rely on different methods like adaptive control [22, 23] or active disturbance rejection [25] in order to compensate for actuator failures. The study of this type of malfunction typically considers either actuators locking in place [22], actuators losing effectiveness but remaining controllable [23, 24], or a combination of both [25]. However, when actuators can be subject to damage or hostile takeover, the malfunction may result in some actuators producing undesirable inputs over which the controller has real-time readings but no control. This type of malfunction has been discussed in [4] under the name of *loss of control authority* over actuators and encompasses scenarios where actuators are under attack [8].

In the setting of loss of control authority, undesirable inputs are observable and can have a magnitude similar to the controlled inputs, while in classical robust control the undesirable inputs are not observable and have a small magnitude compared to the actuators' inputs [12]. The results of [3] showed that a controller having access to the undesirable inputs is considerably more effective than a robust controller.

After a partial loss of control authority over actuators, a target is said to be *resiliently reachable* if for any undesirable inputs produced by the malfunctioning actuators there exists a control driving the state to the target [4]. However, after the loss of control the malfunctioning system might need considerably more time to reach its target compared to the initial system. In this work we thus introduce the concept of quantitative resilience for control systems in order to measure the delays caused by the loss of control authority over actuators. While concepts of quantitative resilience have been previously developed for water infrastructure systems [17] or for nuclear power plants [11], such concepts only work for their specific application.

In this work we formulate quantitative resilience as the maximal ratio of the minimal times required to reach any target for the initial and malfunctioning systems. This formulation leads to a nonlinear minimax optimization problem with an infinite number of equality constraints. Because of the complexity of this problem, a straightforward attempt at a solution is not feasible. While for linear minimax problems with a finite number of constraints the optimum is reached on the boundary of the constraint set [16], such a general result does not hold in the setting of semi-infinite programming [10] where our problem belongs. However, the theorems of [14, 15] stating the existence of time-optimal controls

*Supported by an Early Stage Innovations grant from NASA's Space Technology Research Grants Program, grant no. 80NSSC19K0209. This material is partially based upon work supported by the United States Air Force AFRL/SBRK under contract no. FA864921P0123.

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combined with the specific geometry of our problem, allow us to derive two bang-bang results concerning some nonlinear optimization problems. Then, the quantitative resilience of a driftless system is reduced to single linear optimization problem.

As a first step toward the study of quantitative resilience for linear systems we restrict this work to driftless systems. Indeed, we will see that even with these simple dynamics the theory is already sufficiently rich. Furthermore, one can find an abundance of driftless systems in robotics [18].

The contributions of this paper are fourfold. First, we introduce the concept of quantitative resilience for systems enduring a loss of control authority over some of their actuators. Secondly, to solve our central problem, we determine a simple analytical solution to a related nonlinear optimization problem with applications not restricted only to control theory. Thirdly, we provide an efficient method to compute the quantitative resilience of driftless systems by simplifying a nonlinear problem of four nested optimizations into a single linear optimization problem. Finally, based on quantitative resilience and controllability we establish a necessary and sufficient condition for a system to be resilient.

The remainder of the paper is organized as follows. Section 2 introduces preliminary results concerning resilient systems and defines quantitative resilience. Section 3 establishes two optimization results that will prove crucial for the computation of quantitative resilience. To evaluate this metric we need the minimal time for the system to reach a target before and after the loss of control authority. We calculate this minimal time for the initial system in Section 4 and for the malfunctioning system in Section 5. Section 6 is the pinnacle of this work as we design an efficient method to compute quantitative resilience and assess whether a system is resilient. Finally, in Section 7 our theory is applied to an opinion dynamics scenario.

Notation: The interior of a set X is denoted X° and its convex hull is $co(X)$. Set X is symmetric if $x \in X$ implies $-x \in X$. Let $\mathbb{R}^+ := [0, \infty)$ and we use the subscript $_*$ to exclude zero, for instance $\mathbb{R}^+_* := (0, \infty)$. In $\mathbb{R}^n$ we denote the Euclidean norm with $\|\cdot\|$ and the unit sphere with $\mathbb{S} := \{x \in \mathbb{R}^n : \|x\| = 1\}$. The infinity-norm of $x \in \mathbb{R}^n$ is $\|x\|_\infty := \max\{|x_i| : 1 \leq i \leq n\}$. For integrable piecewise continuous functions $f : \mathbb{R}^+ \to \mathbb{R}^n$, the $\mathcal{L}_2$-norm is defined as $\|f\|^2_{\mathcal{L}_2} := \int_{t \geq 0} \|f(t)\|^2 \, dt$, and the $\mathcal{L}_\infty$-norm is $\|f\|_{\mathcal{L}_\infty} := \sup\big\{\|f(t)\|_\infty : t \geq 0\big\}$.

2 Preliminaries and Problem Statement

We consider driftless systems governed by the dynamics

$$\dot{x}(t) = \bar{B}\bar{u}(t), \qquad x(0) = x_0 \in \mathbb{R}^n, \qquad \bar{u} \in \bar{U}, \tag{2.1}$$

where $\bar{B} \in \mathbb{R}^{n \times (m+p)}$. Even though such systems are seemingly simple, they are used in a variety of applications [18, 19], and we will show that they yield a rich theory of quantitative resilience. The set of allowable controls is

$$\bar{U} := \big\{\bar{u} : \mathbb{R}^+ \to \mathbb{R}^{m+p} : \|u\|_{\mathcal{L}_\infty} \leq u_{\max}\big\}, \tag{2.2}$$

with $u_{max} > 0$. After a malfunction, the system loses control authority over p of its $m+p$ actuators. Because of the malfunction the initial control input $\bar{u}$ is split into the remaining controlled inputs u and the undesirable inputs w. Without loss of generality we consider the columns C representing the malfunctioning actuators to be at the end of $\bar{B}$. We split the control matrix accordingly: $\bar{B} = \big[B \; C\big]$. Then, the dynamics become

$$\dot{x}(t) = Bu(t) + Cw(t), \qquad x(0) = x_0 \in \mathbb{R}^n, \tag{2.3}$$

with $u \in U$, $w \in W$, and

$$\begin{aligned} U &:= \big\{u : \mathbb{R}^+ \to \mathbb{R}^m : \|u\|_{\mathcal{L}_\infty} \leq u_{max}\big\} \\ W &:= \big\{w : \mathbb{R}^+ \to \mathbb{R}^p : \|w\|_{\mathcal{L}_\infty} \leq u_{max}\big\}. \end{aligned} \tag{2.4}$$

We will use the concept of *controllability* of [14].

Definition 2.1. *A system following* (2.1) *is controllable if for all target $x_{goal} \in \mathbb{R}^n$ there exists a control $\bar{u} \in \bar{U}$ and a time T such that $x(T) = x_{goal}$.*

We recall here the definition of the *resilience* of a system introduced in [3].

Definition 2.2. *A system following* (2.1) *is resilient to the loss of p of its actuators corresponding to the matrix C as above, if for all undesirable inputs $w \in W$ and all target $x_{goal} \in \mathbb{R}^n$ there exists a control $u \in U$ and a time T such that the state of the system* (2.3) *reaches the target at time T, i.e., $x(T) = x_{goal}$.*

In previous work [3, 4] the input magnitude was constrained with $\mathcal{L}_2$ bounds, while here we use instead $\mathcal{L}_\infty$ bounds for application purposes. Thus, most of the resilience conditions of [3, 4] do not apply here. We establish a simple necessary condition for our setting.

Proposition 2.1. *If the system* (2.1) *is resilient to the loss of p actuators, then the system $\dot{x}(t) = Bu(t)$ is controllable.*

The proof of this result can be found in the extended version [5] of this paper. By definition, a resilient system is still capable of reaching any target after losing control authority over p of its actuators. However, the time for this malfunctioning system to reach a target might be considerably larger than the time needed for the initial system to reach the same target. We introduce these two times for the target $x_{goal} \in \mathbb{R}^n$ and the target distance $d := x_{goal} - x_0 \in \mathbb{R}^n$.

Definition 2.3. *The* nominal reach time T_N^* *is the shortest time required to reach the target for the initial system following* (2.1)*:*

$$T_N^*(d) := \inf_{\bar u \in \bar U} \Big\{ T \geq 0 : \int_0^T \bar B \bar u(t)\, dt = d \Big\}. \tag{2.5}$$

Definition 2.4. *The* malfunctioning reach time T_M^* *is the shortest time required to reach the target for the malfunctioning system following* (2.3) *when the undesirable input is chosen to make that time the longest:*

$$T_M^*(d) := \sup_{w \in W} \left\{ \inf_{u \in U} \Big\{ T \geq 0 : \int_0^T Bu(t) + Cw(t)\, dt = d \Big\} \right\}. \tag{2.6}$$

By definition, if the system is controllable, then $T_N^*(d)$ is finite for all $d \in \mathbb{R}^n$, and if it is resilient, then $T_M^*(d)$ is finite.

Definition 2.5. *The* ratio of reach times *in the direction* $d \in \mathbb{R}^n$ *is* $t(d) := \frac{T_M^*(d)}{T_N^*(d)}$.

After the loss of control, the malfunctioning system can take up to $t(d)$ times longer than the initial system to reach the target $d + x_0$. Since the performance is degraded by the undesirable inputs, $t(d) \geq 1$. We take the convention that $t(d) = +\infty$ whenever $T_M^*(d) = +\infty$, regardless of the value of $T_N^*(d)$.

Remark 2.1. *The case* $T_N^*(d) = T_M^*(d) = 0$ *can only happen when* $d = 0$, *because* $x(0) = x_0 = x_{goal}$. *We take the convention that* $\frac{T_N^*(0)}{T_M^*(0)} = 1$.

To measure how a system endures a loss of control over its actuators we define its *quantitative resilience.*

Definition 2.6. *The* quantitative resilience r_q *of a system following* (2.3) *is the inverse of the maximal ratio of reach times, i.e.,*

$$r_q := \frac{1}{\sup\limits_{d \in \mathbb{R}^n} t(d)} = \inf_{d \in \mathbb{R}^n} \frac{T_N^*(d)}{T_M^*(d)}. \tag{2.7}$$

Quantitative resilience can be defined in exactly the same way for general control systems, but we focus on linear driftless systems in this work. For a resilient system, $r_q \in (0, 1]$. The closer r_q is to 1, the smaller is the loss of performance caused by the malfunction.

Quantitative resilience r_q depends on matrices B and C, i.e., on the actuators that are producing undesirable inputs. For a system following (2.1), one can calculate r_q for all possible malfunctions. Computing r_q directly from (2.5)-(2.7) requires solving four nested optimization problems, with three constraint sets being infinite-dimensional function spaces. A brute force approach to this problem is doomed to fail. Thus, we focus on the following problem.

Problem 1. *How to compute* r_q *efficiently?*

3 Optimization on Polytopes

In this section, we introduce two novel optimization results on polytopes that will be needed to compute quantitative resilience. To save space, full proofs of these results can be found in the extended paper [5].

Definition 3.1. *A* polytope *in* $\mathbb{R}^n$ *is a compact intersection of finitely many half-spaces.*

Definition 3.2. *A* vertex *of a set* $X \subset \mathbb{R}^n$ *is a point* $x \in X$ *such that if there are* $x_1 \in X$ *and* $x_2 \in X$ *with* $x \in [x_1, x_2]$, *then* $x = x_1 = x_2$.

With these definitions, polytopes are convex, and a vertex of a polytope corresponds to the usual understanding of a vertex of a polytope.

Theorem 3.1. *Let* $d \in \mathbb{R}^n_*$, X *and* Y *two polytopes in* $\mathbb{R}^n$ *with* $-X \subset Y$. *Then, there exists* v *a vertex of* X *such that* $\|v + y^*(v)\| = \min\limits_{x \in X} \|x + y^*(x)\|$, *with* $y^*(x) := \arg\max\limits_{y \in Y} \big\{ \|x + y\| : x + y \in \mathbb{R}^+ d \big\}$.

Theorem 3.1 will help us calculate the malfunctioning reach time T_M^* of resilient systems, while the following result will simplify the calculation of r_q.

Theorem 3.2. *If* X *and* Y *are two symmetric polytopes in* $\mathbb{R}^n$ *with* $X \subset Y^\circ$, $\dim X = 1$, $\partial X = \{x, -x\}$ *and* $\dim Y = n$, *then* $\max\limits_{d \in \mathbb{S}} r_{X,Y}(d) = r_{X,Y}(x)$, *with*

$$r_{X,Y}(d) := \frac{\max\limits_{x \in X,\, y \in Y} \big\{ \|x + y\| : x + y \in \mathbb{R}^+ d \big\}}{\min\limits_{x \in X} \big\{ \max\limits_{y \in Y} \big\{ \|x + y\| : x + y \in \mathbb{R}^+ d \big\} \big\}}. \tag{3.8}$$

Proof. In [5], we prove the existence of the max and min in (3.8) with the compactness of X and Y. Geometric arguments then show that x maximizes $r_{X,Y}$. □

We now return to the discussion of resilient systems.

4 Dynamics of the Initial System

We start with the initial system of dynamics (2.1) and aim to calculate the nominal reach time T_N^*. We introduce $\bar U_c := \big\{ \bar u \in \mathbb{R}^{m+p} : \|\bar u\|_\infty \leq u_{max} \big\}$, the set of constant inputs.

Proposition 4.1. *For a controllable system* (2.1) *and* $d = x_{goal} - x_0 \in \mathbb{R}^n$, *the infimum* $T_N^*(d)$ *of* (2.5) *is achieved with a constant control input* $\bar u^* \in \bar U_c$.

Proof. Dynamics (2.1) are linear in x and $\bar u$. Set $\bar U$ defined in (2.2) is convex and compact. The system is controllable, so x_{goal} is reachable. The assumptions of Theorem 4.3 of [14] are satisfied, leading to the existence of a time optimal control $\hat{u} \in \bar{U}$. Thus, the infimum in (2.5) is a minimum and $\int_0^{T_N^*} \bar{B}\hat{u}(t)\,dt = d$. If $d = 0$, then, according to Remark 2.1, $T_N^* = 0$ and we take $\bar{u}^* = 0 \in \bar{U}_c$. Otherwise, $T_N^* > 0$, so we can define the vector $\bar{u}^* := \frac{1}{T_N^*}\int_0^{T_N^*} \hat{u}(t)\,dt \in \mathbb{R}^{m+p}$. Since $\hat{u} \in \bar{U}$ we have $\bar{u}^* \in \bar{U}_c$ and $\int_0^{T_N^*} \bar{B}\bar{u}^*\,dt = \bar{B}\bar{u}^* T_N^* = d$. ☐

Because of Proposition 4.1, (2.5) simplifies to

$$T_N^*(d) = \min_{\bar{u} \in \bar{U}_c} \big\{ T \geq 0 : \bar{B}\bar{u}\,T = d \big\}. \tag{4.9}$$

The multiplication of the variables $\bar{u}$ and T prevents the use of linear solvers for (4.9). Instead, we will consider

$$T_N^*(d) = \left(\max_{\bar{u} \in \bar{U}_c} \big\{ \lambda : \bar{B}\bar{u} = \lambda d \big\} \right)^{-1}, \tag{4.10}$$

after using the transformation $\lambda = \frac{1}{T}$ in (4.9). Problem (4.10) is linear in $\bar{u}$ so the optimal control input $\bar{u}^*$ belongs to the boundary of the constraint set [16] for $d \neq 0$. Thus, $\|\bar{u}^*\|_\infty = u_{max}$.

PROPOSITION 4.2. *The nominal reach time T_N^* is absolutely homogeneous, i.e., $T_N^*(\lambda d) = |\lambda|\ T_N^*(d)$ for $d \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$.*

Proof. For $\lambda = 0$, we have $T_N^*(0) = 0$. Let $\lambda > 0$ and $d \in \mathbb{R}^n$. From (4.9), there is $\bar{u}_d \in \bar{U}_c$ so that $\bar{B}\bar{u}_d T_N^*(d) = d$, and $\bar{B}\,\bar{u}_d\,\lambda T_N^*(d) = \lambda d$. The optimality of $T_N^*(\lambda d)$ to reach λd leads to $T_N^*(\lambda d) \leq \lambda T_N^*(d)$.

There exists $\bar{u}_{\lambda d} \in \bar{U}_c$ such that $\bar{B}\bar{u}_{\lambda d} T_N^*(\lambda d) = \lambda d$. Then $\bar{B}\,\bar{u}_{\lambda d}\,\frac{T_N^*(\lambda d)}{\lambda} = d$. The optimality of $T_N^*(d)$ to reach d leads to $T_N^*(d) \leq \frac{T_N^*(\lambda d)}{\lambda}$, so $\lambda T_N^*(d) = T_N^*(\lambda d)$. For $\lambda < 0$, the proof is similar to the above, and for all details see [5]. ☐

We can now tackle the dynamics of the malfunctioning system after a loss of control authority over some of its actuators.

5 Dynamics of the Malfunctioning System

We study the system of dynamics (2.3) to compute the malfunctioning reach time T_M^*. We define the constant input sets $U_c := \big\{u \in \mathbb{R}^m : \|u\|_\infty \leq u_{max}\big\}$, $W_c := \big\{w \in \mathbb{R}^p : \|w\|_\infty \leq u_{max}\big\}$, and V_c the set of vertices of W_c.

PROPOSITION 5.1. *For a resilient system, $d \in \mathbb{R}^n_*$ and $w \in W$, the infimum $T_M(w,d)$ of (2.6) defined as $T_M(w,d) = \inf_{u \in U} \Big\{ T \geq 0 : \int_0^T Bu(t) + Cw(t)\,dt = d \Big\}$, is achieved with a constant control input $u_d^*(w) \in U_c$.*

The proof of Proposition 5.1 is similar to that of Proposition 4.1, and can be found in [5] to save space. We can now work on the supremum of (2.6).

PROPOSITION 5.2. *For a resilient system and $d \in \mathbb{R}^n_*$, the supremum $T_M^*(d)$ of (2.6) is achieved with a constant undesirable input $w^* \in W_c$.*

Proof. Let $w \in W$, $d \in \mathbb{R}^n_*$, $w_c := \int_0^{T_M(w,d)} \frac{w(t)dt}{T_M(w,d)}$, with $T_M(w,d)$ from Proposition 5.1. Then, $w_c \in W_c$ and $Bu_d^*(w)T + \int_0^T Cw(t)\,dt = d = \big(Bu_d^*(w) + Cw_c\big)T$. Thus, the supremum of (2.6) can be taken on W_c.

We define the function $\varphi : W_c \to \mathbb{R}^n$ as

$$\varphi(w_c) := Bu_d^*(w_c) + Cw_c \qquad \text{for} \quad w_c \in W_c. \tag{5.11}$$

When applying w_c and $u_d^*(w_c)$ the dynamics become $\dot{x} = \varphi(w_c)$. We prove in [5] that φ is continuous in w_c. Set W_c is compact, $t_0 = 0$ and $x_0 \in \mathbb{R}^n$ are fixed. Then, Theorem 1 of [15] states that the attainable set $\mathcal{A}_{W_c} := \big\{(x_1, T) : \text{for } w_c \in W_c, \int_0^T \varphi(w_c)\,dt = x_1 - x_0\big\}$ is compact. Since $T_M^*(d) = \sup\big\{T : (x_{goal}, T) \in \mathcal{A}_{W_c}\big\}$, the supremum of (2.6) is achieved on W_c. ☐

Following Propositions 5.1 and 5.2, (2.6) becomes

$$T_M^*(d) = \max_{w_c \in W_c} \left\{ \min_{u_c \in U_c} \big\{ T \geq 0 : \big(Bu_c + Cw_c\big)T = d \big\} \right\}. \tag{5.12}$$

The simplifications achieved so far were based on existence theorems from [14, 15] upon which the bang-bang principle relies. The logical next step is to show that the maximum of (5.12) is achieved on a vertex of W_c. However, most of the work on the bang-bang principle considers systems with a linear dependency on the input [13, 14, 21], while φ introduced in (5.11) is not linear.

The work [15] considers a nonlinear φ, yet the discussion on bang-bang inputs would require us to show that $co(\varphi(W_c)) = co(\varphi(V_c))$. This task is not trivial as it amounts to proving that inputs in V_c can do as much as inputs in W_c, i.e., we would need to prove the bang-bang principle. Two more works [1, 9] study bang-bang properties of systems with nonlinear dependency on the input. However, both of them require conditions that are not met in our case. The results of [1] require $\dot{x} = Cw$ to be controllable, while [9] needs $T_M(w,d)$ to be concave in w. Thus, even if bang-bang theory seems like a natural approach, we had to establish our own optimization result, namely Theorem 3.1, in order to show that the maximum of (5.12) is achieved on V_c.

PROPOSITION 5.3. *For a resilient system and $d \in \mathbb{R}^n_*$, the maximum of (5.12) is achieved with a constant input $w^* \in V_c$, i.e., its components are $\pm u_{max}$.*

Proof. We first introduce the two polytopes of $\mathbb{R}^n$, $X := \{Cw_c : w_c \in W_c\}$ and $Y := \{Bu_c : u_c \in U_c\}$. Then, using $\lambda = \frac{1}{T}$ in (5.12) we have

$$\frac{1}{T_M^*(d)} = \min_{x \in X} \big\{ \max_{y \in Y} \{ \lambda \geq 0 : x + y = \lambda d \} \big\}.$$

Since $\lambda \geq 0$, we can write $\lambda = |\lambda| = \frac{\|\lambda d\|}{\|d\|} = \frac{\|x+y\|}{\|d\|}$. Then, our problem of interest becomes

$$\frac{1}{\|d\|} \min_{x \in X} \Big\{ \max_{y \in Y} \{ \|x + y\| : x + y \in \mathbb{R}^+ d \} \Big\}. \tag{5.13}$$

To apply Theorem 3.1, we need to show that $-X \subset Y$. Since the system is resilient, for all $w_c \in W_c$ and all $d_0 \in \mathbb{R}^n$ there exists $u_c \in U_c$ and $T \geq 0$ such that $(Bu_c + Cw_c)T = d_0$. Then, for $x = Cw_c \in X$, $x \neq 0$ and $d_0 = -x$ there exists $y \in Y$ and $T > 0$ such that $(x + y)T = -x$. Then, $y = -\lambda x$ with $\lambda := 1 + 1/T > 1$. Since $0 \in Y$ and $-\lambda x \in Y$ then $-x \in Y$ by convexity of Y. Thus, $-X \subset Y$.

We can now apply Theorem 3.1 and conclude that the minimum x^* of (5.13) must be realized on a vertex of X. In [5] we prove that there exists $v^* \in V_c$ such that $x^* = Cv^*$. ∎

We have reduced the constraint set of (2.6) from an infinite-dimensional set W to a finite set V_c of cardinality 2^p, with p being the number of malfunctioning actuators. Then,

$$T_M^*(d) = \max_{w_c \in V_c} \Big\{ \min_{u_c \in U_c} \{ T \geq 0 : (Bu_c + Cw_c)T = d \} \Big\}. \tag{5.14}$$

Similarly to the nominal reach time, T_M^* is also linear in the target distance.

PROPOSITION 5.4. *The malfunctioning reach time T_M^* is absolutely homogeneous, i.e., $T_M^*(\lambda d) = |\lambda|\, T_M^*(d)$ for $d \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$.*

Proof. Because of the minimax structure of (5.14), scaling like in the proof of Proposition 4.2 is not sufficient for $T_M^*(d)$. The complete proof is more complex as it relies on the continuity of T_M^* and can be found in [5]. ∎

We can now combine the initial and malfunctioning dynamics.

6 Quantitative Resilience

Quantitative resilience is defined in (2.7) as the infimum of $T_N^*(d)/T_M^*(d)$ over $d \in \mathbb{R}^n$. Using Propositions 4.2 and 5.4 we reduce this constraint to $d \in \mathbb{S}$. For the loss of control over a single actuator we can determine the optimal $d \in \mathbb{S}$ by noting that the effects of the undesirable inputs are the strongest along the direction described by the malfunctioning actuator.

THEOREM 6.1. *For a resilient system following (2.3) with C a single column matrix, $\max_{d \in \mathbb{S}} t(d) = t(C)$.*

Proof. Fix $d \in \mathbb{S}$. As in (4.10), we rewrite the malfunctioning reach time

$$\begin{aligned} T_M^*(d) &= \max_{w_c \in W_c} \big\{ \min_{u_c \in U_c} \{ T : (Bu_c + Cw_c)T = d \} \big\} \\ &= \frac{1}{\min_{w_c \in W_c} \big\{ \max_{u_c \in U_c} \{ \lambda : Bu_c + Cw_c = \lambda d \} \big\}}. \end{aligned}$$

Let $Y := \{Bu_c : u_c \in U_c\}$ and $X := \{Cw_c : w_c \in W_c\}$. Since $\lambda \geq 0$ and $\|d\| = 1$, $\lambda = \|\lambda d\| = \|y + x\|$. These simplifications lead to

$$T_M^*(d) = \frac{1}{\min_{x \in X} \big\{ \max_{y \in Y} \{ \|y + x\| : y + x \in \mathbb{R}^+ d \} \big\}}. \tag{6.15}$$

We focus on the nominal reach time and proceed to the separation of $\bar{B} = [B\ C]$ in (4.10):

$$\begin{aligned} \frac{1}{T_N^*(d)} &= \max_{\bar{u} \in \bar{U}_c} \{ \lambda : \bar{B}\bar{u} = \lambda d \} \\ &= \max_{\substack{u_c \in U_c \\ w_c \in W_c}} \{ \lambda : Bu_c + Cw_c = \lambda d \} \\ &= \max_{\substack{y \in Y \\ x \in X}} \{ \|y + x\| : y + x \in \mathbb{R}^+ d \}. \end{aligned} \tag{6.16}$$

We can now gather (6.15) and (6.16) into

$$\begin{aligned} t(d) = \frac{T_M^*(d)}{T_N^*(d)} &= \frac{\max_{x \in X,\, y \in Y} \{ \|x + y\| : x + y \in \mathbb{R}^+ d \}}{\min_{x \in X} \big\{ \max_{y \in Y} \{ \|x + y\| : x + y \in \mathbb{R}^+ d \} \big\}} \\ &= r_{X,Y}(d), \end{aligned}$$

with $r_{X,Y}$ defined in (3.8). In the proof of Proposition 5.3 we used that X and Y are polytopes verifying $-X \subset Y$. Since U_c and W_c are symmetric, so are X and Y. In [5] we show that the resilience of the system implies $X \subset Y^\circ$ and $\dim Y = n$. Because C is a single column, $\dim X = 1$ and $\partial X = \{-Cu_{max}, Cu_{max}\}$. We can then apply Theorem 3.2 and obtain $\max_{d \in \mathbb{S}} t(d) = t(Cu_{max}) = t(C)$ because $u_{max} \in \mathbb{R}$ and t is invariant to scaling according to Propositions 4.2 and 5.4. ∎

Thus, to calculate r_q we only need $T_N^*(C)$ and $T_M^*(C)$. The computation load can be even further reduced with the following result.

Theorem 6.2. *For a resilient system losing control over a single nonzero column C, $r_q = r_{max}$, where*

$$r_{max} := \frac{\lambda^* - u_{max}}{\lambda^* + u_{max}} \quad \text{and} \quad \lambda^* := \max_{v \in U_c} \big\{ \lambda : Bv = \lambda C \big\}. \tag{6.17}$$

Proof. Let $\bar{u} \in \bar{U}_c$, $u \in U_c$ and $w \in V_c \subset \mathbb{R}$ be the arguments of the solutions of (4.9) and (5.14) for $d = C \neq 0$. We split $\bar{u} = (u_B, u_C)$ with $u_B \in U_c$ and $u_C \in W_c \subset \mathbb{R}$. Then,

$$\bar{B}\bar{u}\, T_N^*(C) = \big(Bu_B + Cu_C\big) T_N^*(C) = C, \tag{6.18}$$

$$\big(Bu + Cw\big) T_M^*(C) = C. \tag{6.19}$$

Since C is a single column, Cw and Cu_C are collinear with C. Then, Bu_B and Bu are also collinear with C, so there exists $\lambda_M \in \mathbb{R}$ and $\lambda_N \in \mathbb{R}$ such that $Bu_B = \lambda_N C$ and $Bu = \lambda_M C$. Then, (6.18) and (6.19) become scalar equations

$$\lambda_N + u_C = 1/T_N^*(C), \tag{6.20}$$

$$\lambda_M + w = 1/T_M^*(C). \tag{6.21}$$

Note that λ_M and w are independent. Since $u \in U_c$ must maximize the right-hand side of (6.21), we have $\lambda_M = \max\limits_{u \in U_c} \big\{ \lambda : Bu = \lambda C \big\}$ whatever the value of w. Similarly, λ_N and u_C are independent, which leads to $\lambda_N = \lambda_M = \lambda^*$ as defined in (6.17).

From Proposition 5.3, $w = \pm u_{max}$ and is chosen to minimize the right-hand side of (6.21), so $w = -u_{max}$. On the other hand, u_C must maximize $1/T_N^*(C)$ so $u_C = u_{max}$. Then, (6.20) and (6.21) become

$$\lambda^* + u_{max} = 1/T_N^*(C), \quad \text{and} \quad \lambda^* - u_{max} = 1/T_M^*(C).$$

By Theorem 6.1, $r_q = \frac{T_N^*(C)}{T_M^*(C)} = \frac{\lambda^* - u_{max}}{\lambda^* + u_{max}} = r_{max}$. ∎

We introduced quantitative resilience as the solution of four nonlinear nested optimization problems and now we reduced r_q to the solution of a single linear optimization problem. We have completed Problem 1.

So far, all our results need the system to be resilient. However, the resilience criteria of [3] cannot be applied here due to a difference in the set of allowable controls. We thus establish a new resilience condition proved in the extended version [5] of this paper.

Proposition 6.1. *A system following* (2.1) *is resilient to the loss of control over a column C if and only if it is controllable and $T_M^*(C)$ is finite.*

Controllability guarantees that every state is reachable and $T_M^*(C)$ being finite ensures that the reachability is preserved despite the worst undesirable inputs. However, following Theorem 6.2 we do not need to compute $T_M^*(C)$ to access r_q, so we derive in [5] an equivalent resilience condition involving directly r_{max}.

Corollary 6.1. *A system following* (2.1) *is resilient to the loss of control over a nonzero column C if and only if it is controllable and $r_{max} \in (0, 1]$.*

We now have all the tools to assess the quantitative resilience of a driftless system. If $\bar{B}$ is not full rank, the system following (2.1) is not controllable and there is no need to go further. Otherwise, we compute r_{max} and using Corollary 6.1 we assess whether the system is resilient. If it is, Theorem 6.2 states that $r_q = r_{max}$, otherwise $r_q = 0$. We will now apply this method to an opinion dynamics scenario.

7 Numerical Example: Opinion Dynamics

Opinion dynamics study how a group of agents shape their opinions x in different scenarios, for instance facing outside opinion sources u. Such a situation is illustrated in [19] with a 1D discrete time Deffuant model $x(t+1) = x(t) + \mu\varepsilon\big(u(t) - x(t)\big)$, where μ is a convergence parameter and ε encodes the strength of u. For our purpose, we will consider u as an input to the system and generalize to a multi-inputs, multi-states continuous time model: $\dot{x}(t) = Ax(t) + \bar{B}\bar{u}(t)$. We assume that agents have no direct interactions with each other, leading to the driftless model $\dot{x}(t) = \bar{B}\bar{u}(t)$. Similar models are used in consensus dynamics [6] where $\bar{B} = I$ and controls are state feedback $\bar{u}(t) = -Kx(t)$.

We refer to the outside sources as *channels*. An example is a consumer of multiple media sources with different levels of trust towards different media. The agents opinions are solely determined by the controller of the channels.

The controller is using its channels to steer the opinion of each agent towards a target set. For instance, the controller could be a worldwide media conglomerate such as the News Corporation [2]. The COVID-19 pandemic has minimized direct interactions between people, hence making our setting more realistic. An extreme variant of this scenario is illustrated by the episode "Fifteen Million Merits" of the Black Mirror series [7].

A perturbing event, e.g., loss of influence, foreign acquisition of a news channel, or a new board of directors, causes one of the channels to become uncontrollable and to produce undesirable inputs. The controller has still access to this channel and is informed in real time of its content, while being unable to modify it.

We consider $n = 3$ agents having initially a neutral opinion: $x_0 = (0,\ 0,\ 0)$. Then, the target is $d = x_{goal} \in \mathbb{R}^3$. For instance, $d = (1,\ 1,\ 1)$ is a *consensus target*, while $d = (-1,\ -1,\ 1)$ is a *polarization target*. The components of $\bar{B}$, denoted by $\bar{B}_{i,j} \in [-1, 1]$ reflect the influence of channel j over agent i. We consider 6 different channels:

$$\bar{B} = \begin{bmatrix} 0.8 & -0.9 & 0.4 & -0.4 & -0.7 & 0 \\ 1 & -1 & 0.3 & 0.2 & 0.7 & -0.1 \\ 0.9 & -0.8 & -0.4 & -0.4 & 0.7 & 0.1 \end{bmatrix}.$$

Given the elements of $\bar{B}$, the agents strongly trust channel 1 but not channel 2, they have diverging moderate opinions on channels 3 and 4, strongly diverging trust of channel 5, while they are barely influenced by channel 6. We compute r_{max} for the loss of control over each single channel:

$$r_{max} = \begin{bmatrix} 0.20 & 0.18 & 0.15 & 0.19 & -0.22 & 0.75 \end{bmatrix}.$$

Since $\bar{B}$ is full rank and only $r_{max}(5) \notin (0, 1]$, according to Corollary 6.1, the system is resilient to the loss of control over any single channel except channel 5. Using Theorem 6.2, for channel 5 we have $r_q(5) = 0$, while $r_q(i) = r_{max}(i)$ for the other channels i. Because none of the agents are significantly influenced by channel 6, the quantitative resilience to its loss is the greatest.

The inverse of r_q describes the factor by which the time to reach an opinion state may increase after a malfunction:

$$\frac{1}{r_q} = \begin{bmatrix} 4.9 & 5.5 & 6.5 & 5.4 & \infty & 1.3 \end{bmatrix}. \tag{7.22}$$

From Theorem 6.1, we know that $\frac{1}{r_q(j)} = t(\bar{B}_j)$, which is the ratio of reach times in the direction $\bar{B}_j$, the j^{th} column of $\bar{B}$. Thus, if $x_{goal} = \bar{B}_3 = (0.4,\ 0.3,\ -0.4)$, then after the loss of control over channel 3, there exists an undesirable input causing an increase of the time to reach this target by a factor 6.5. On the other hand, the loss of control over channel 6 has a much smaller impact on the time to reach any target.

We now choose the target $x_{goal} = d = (1,\ 1,\ 1)$ and compare how the loss of each channel affects the delay to reach this target. Intuitively, since x_{goal} is a consensus target, losing control over the channel 1 or 2 will have a considerable impact, while the loss of the other (channels except 5) should not be significant. Indeed, when calculating $t(d)$ for the loss of each channel we obtain

$$t(d) = \begin{bmatrix} 4.8 & 5.4 & 1.6 & 1.6 & \infty & 1.0 \end{bmatrix},$$

which confirms our intuition.

If the controller has polarization objectives, for instance $d = (-1,\ -1,\ 1)$, then losing control of channel 3, 4 or 5 should be problematic, while the others should have a smaller impact. Indeed,

$$t(d) = \begin{bmatrix} 1.7 & 1.7 & 6.5 & 5.4 & \infty & 1.3 \end{bmatrix},$$

so the loss of channel 5 renders the polarization target unreachable, and loss of channel 3 or 4 causes a large delay.

To illustrate the proof of Theorem 3.2 we compute the ratio $t(d) = r_{X,Y}(d)$ while completing a revolution with $d \in \mathbb{S} \subset \mathbb{R}^3$ for the loss of channel 1. The red spikes show when d is collinear with C. As can be seen on Figure 1, the spikes coincide with the maximum of $t(d)$, located at 4.9 as announced in (7.22). We also note that $t = r_{X,Y}$ is an even function of d as proven in [5].

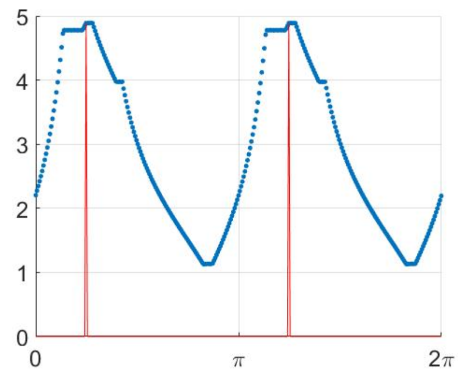

Figure 1: Ratio of reach times $t(d)$ for the loss of control over channel 1.

8 Conclusion and Future Work

To quantify the drop in performance caused by the loss of control authority over actuators, this paper introduced the notion of quantitative resilience for control systems. Relying on bang-bang control theory and on two novel optimization results, we transformed a nonlinear problem consisting of four nested optimizations into a single linear problem. This simplification leads to a computationally efficient algorithm to verify resilience and calculate the quantitative resilience of driftless systems.

There are three promising avenues of future work. To study systems like drones, whose inputs are only positive propeller velocities, we need to extend our theory to asymmetric input sets. Secondly, we have only considered driftless systems because of the complexity of the subject. However, future work should be able to extend the concept of quantitative resilience to non-driftless linear systems. Finally, noting that Theorems 6.1 and 6.2 only concern the loss of a single actuator, our third direction of work is to extend these results to the simultaneous loss of multiple actuators.

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