A Topological Obstruction in a Control Problem

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Abstract

One of the important discoveries in control theory is a topological obstruction to continuous feedback stabilization for general nonlinear control systems. In this note we describe another topological obstruction arising from a very different control problem called the *reach control problem*. Motivated by a classical topological obstruction for extending continuous maps on spheres, we introduce the problem of extending continuous maps on simplices. It is shown that the same condition as in the sphere case gives rise to the obstruction.

Keywords: reach control problem, topological obstruction, extension problem

1. Introduction

This paper regards the Reach Control Problem (RCP). The RCP seeks to find a feedback control which drives the trajectories of an affine system initialized in a simplex to reach and exit a facet of the simplex. We direct the reader to [14, 27] and other cited works which regard the RCP for a more substantial discussion. For the purposes of this introduction, it suffices to state that this line of research is motivated by the desire to satisfy complex control specifications given in a constrained state space; this research thrust is also present, for instance, in [10, 11, 19], although the settings and methods of those papers are vastly different from the reach control setting.

The classical theory of controllability is largely focused on control strategies in the Euclidean space, and the methods for dealing with control theory in constrained spaces are still under development. While we make no claim to provide an extensive discussion of such efforts within this paper, we direct the reader to a vast discussion and body of references in [2]. We additionally note that this work is related in spirit to results of [12, 13, 16, 17, 18], as well as other

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works which deal with controllability or reachability of systems with constraints. In the interest of space we will not delve into a further discussion of these papers. While the results in the current paper are motivated by similar lines of inquiry as the previously mentioned works, the theoretical results obtained and methods used to do so are invariably vastly different from the current paper. We point the reader to those papers and the references contained therein for more information on the control of systems with state and input constraints. In this paper, we investigate an obstruction to solving the RCP via continuous state feedback. Noteworthy is the analogy with the topological obstruction to continuous feedback stabilization [5, 9].

The RCP has been extensively studied with emphasis on finding a complete class of controls to solve the problem [6, 1, 7]. Affine state feedback has played the dominant role, in analogy with linear state feedback to solve the stabilization problem for linear systems [14, 27]. Under a special triangulation of the state space, it has been shown that affine feedback and continuous state feedback are equivalent with respect to solving the RCP [6]. Under the same triangulation, piecewise affine feedback or time-varying affine feedback may be used when continuous state feedbacks fail [7, 1]. However, determining a set of easily verifiable sufficient and necessary conditions for the solvability of the RCP under continuous state feedback solving the RCP has thus far remained elusive. The recent research effort on the topic of a topological obstruction is an attempt to obtain easily verifiable strong necessary, albeit not sufficient, condition for the solvability of the RCP.

Our investigation of a topological obstruction has precursors in two specific areas. The initial piece of motivation is given by [7], where we investigated a situation when continuous state feedback fails under a special triangulation. It was discovered that the failure arises from two conditions. First, the control system is underactuated, meaning there are not sufficient control inputs to resolve the requirements of the RCP. Second, available control directions are not adapted to the simplex so that even with high gain control, closed-loop equilibria appear in the simplex using continuous state feedback, resulting in a failure to solve the problem. The main tool to prove existence of equilibria was Sperner's lemma [6]. We use a similar proof method here. This paper can be regarded as a generalization of [7] to the case of arbitrary triangulations.

The second precursor of this investigation is contained in [28, Theorem 1]. It identifies a cone condition relating the available control directions to the geometry of the simplex as a necessary condition to solve the RCP by continuous state feedback. The result was for single input systems only. Due to its reliance on the intermediate value theorem, the proof cannot be easily extended to systems with multiple inputs. The same cone condition again emerges in the present work; however, now we work with multi-input systems.

The topological obstruction has already been investigated in [21, 22, 26, 25]. We now clarify the differences between the present paper and our previous work. All three above papers use topological methods for establishing sufficient and necessary conditions for the existence of a topological obstruction to the solvability of RCP. The paper [22] treats the case of two- and three-dimensional simplices, [21] discusses systems with two inputs, and [26] and [25] assume that an underlying affine system is controllable, and use that assumption to provide the most general currently available characterization of the topological obstruction problem. In this paper, we assume a special symmetric structure on the set of possible equilibria in the simplex, which does not appear as a requirement in either of the three above papers. This structure enables us to use algebraic methods to show that the cone condition identified in [28] is a necessary and sufficient condition for the existence of a topological obstruction, under the assumptions of this paper. This condition is significantly simpler than the conditions discovered in previous papers, at the expense of more restrictive assumptions. The proof methods are also different. Specifically, [22] heavily uses retractions, while [21], [26], and [25] use homotopy and extension theory. In this paper, topological methods are mostly limited to our work on the sphere, while the obstruction on a simplex is investigated using linear algebra and the Knaster-Kuratowski-Mazurkiewicz (KKM) lemma.

We also note the recent work of [23] and [24], which opens the space for connecting the problem of a topological obstruction to wider results in control theory. In [23], the problem of the solvability of the RCP is related to the notion of local controllability. However, this connection has not yet been formalized. More significantly, [24] poses the RCP in terms of a problem on positive systems. Hence, the results presented in this paper can be interpreted as results on the existence of equilibria in the setting of positive systems. This will be further briefly discussed in Section 2.

The contributions of the paper are as follows. In Section 4, we formulate the problem of a topological obstruction on the sphere as an analogue to the known question of a topological obstruction to the RCP on a simplex. Using degree theory, that problem is then solved at the end of Section 4. In Section 5, we go back to the problem of a topological obstruction on a simplex. We solve the problem under certain additional assumptions on the structure of the set of possible equilibria.

Notation. Let $\mathcal{X}$ and $\mathcal{Y}$ be sets. The direct sum of $\mathcal{X}$ and $\mathcal{Y}$ is denoted by $\mathcal{X} \oplus \mathcal{Y}$. If $\mathcal{X}$ is contained in a topological space, notation $\mathcal{X}^\circ$ denotes the (relative) interior of $\mathcal{X}$, while $\overline{\mathcal{X}}$ denotes its (relative) closure, and $\partial\mathcal{X}$ denotes its (relative) boundary. Notation $id_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}$ represents the identity map. The symbol for an n-dimensional unit ball is $\mathbb{B}^n$, and for an $(n-1)$-dimensional unit sphere is $\mathbb{S}^{n-1}$. Notations $\text{co}\{v_1, \dots, v_k\}$ and $\text{span}\{v_1, \dots, v_k\}$ denote the convex hull and vector subspace generated by points $v_1, \dots, v_k$, respectively.

2. Reach Control Problem

In this section we introduce the control problem which gives rise to our study of a topological obstruction. We consider an n-dimensional simplex $\mathcal{S} := \text{co}\{v_0, \dots, v_n\}$, the convex hull of $n+1$ affinely independent points in $\mathbb{R}^n$. Let its vertex set be $V := \{v_0, \dots, v_n\}$ and its facets $\mathcal{F}_0, \dots, \mathcal{F}_n$. The facet will be indexed by the vertex it does not contain. Equivalently, the facet $\mathcal{F}_i$ is the convex hull of vertices $v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n$. Without loss of generality, we assume that $v_0 = 0$. Let h_j, $j \in \{0, \ldots, n\}$, be the unit normal vector to each facet $\mathcal{F}_j$ pointing outside of the simplex. Facet $\mathcal{F}_0$ is called the *exit facet*. Let $I := \{1, \ldots, n\}$ and define $I(x)$ to be the minimal index set among $\{0, \ldots, n\}$ such that $x \in \mathrm{co}\{v_i \mid i \in I(x)\}$. For $x \in \mathcal{S}$ define the closed, convex cone

$$\mathcal{C}(x) := \left\{ \ y \in \mathbb{R}^n \mid h_j \cdot y \leq 0, \ j \in I \setminus I(x) \ \right\}. \tag{2.1}$$

(Note that h_0 never appears and $\mathcal{C}(x) = \mathbb{R}^n$ for $x \in \mathcal{S}^\circ$.) Figure 1, modified from [20], illustrates our notation for a 2D simplex.

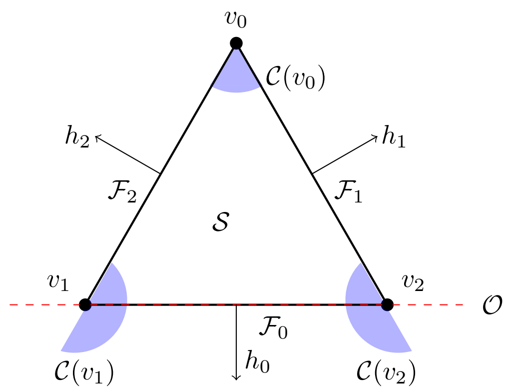

Figure 1: A simplex $\mathcal{S} = \mathrm{co}\{v_0, v_1, v_2\}$ with vertices $V = \{v_0, v_1, v_2\}$ and facets $\mathcal{F}_0$, $\mathcal{F}_1$, and $\mathcal{F}_2$. The unit normal vector of each $\mathcal{F}_i$ pointing out of $\mathcal{S}$ is h_i. The cones $\mathcal{C}(v_i)$ are shown attached at each v_i.

We consider the affine control system on $\mathcal{S}$:

$$\dot{x} = Ax + Bu + a\,, \qquad\qquad x \in \mathcal{S}\,, \tag{2.2}$$

where $A \in \mathbb{R}^{n \times n}$, $a \in \mathbb{R}^n$, $B \in \mathbb{R}^{n \times m}$, and $\mathrm{rank}(B) = m$. Let $\mathcal{B} = \mathrm{Im}(B)$, the image of B. Let $\phi_u(t, x_0)$ denote the trajectory of (2.2) starting at x_0 under control input u. We are interested in studying reachability of the exit facet $\mathcal{F}_0$ from $\mathcal{S}$.

Problem 1 (Reach Control Problem (RCP))**.** *Consider system (2.2) defined on $\mathcal{S}$. Find a feedback $u(x)$ such that: for each $x_0 \in \mathcal{S}$ there exist $T \geq 0$ and $\delta > 0$ such that*

(i) $\phi_u(t, x_0) \in \mathcal{S}$ *for all* $t \in [0, T]$,

(ii) $\phi_u(T, x_0) \in \mathcal{F}_0$, *and*

(iii) $\phi_u(t, x_0) \notin \mathcal{S}$ *for all* $t \in (T, T + \delta)$.

The RCP says that trajectories of (2.2) starting from initial conditions in $\mathcal{S}$ exit $\mathcal{S}$ through the exit facet $\mathcal{F}_0$ in finite time, while not first leaving $\mathcal{S}$. In order for a feedback $u(x)$ to solve Problem 1, $Ax + Bu + a$ cannot have any equilibria in $\mathcal{S}$. We observe that $Ax + Bu + a$ can vanish for an appropriate choice of u only if $x \in \mathcal{O}$ where $\mathcal{O} := \{ x \in \mathbb{R}^n \mid Ax + a \in \mathcal{B}\}$. Thus, if $u(x)$ is a continuous state feedback, then equilibria of the closed-loop system can only appear in

$$\mathcal{G} := \mathcal{S} \cap \mathcal{O} .$$

Additionally, to solve the RCP we require conditions that disallow trajectories to exit from the facets $\mathcal{F}_i$, $i \in I$. We say that a state feedback $u(x)$ satisfies the *invariance conditions* if

$$Ax + Bu(x) + a \in \mathcal{C}(x) \, , \; x \in \mathcal{S} . \tag{2.3}$$

The invariance conditions are necessary conditions to solve the RCP [7].

We wish to investigate when there exists a continuous $u(x)$ satisfying (2.3) such that there are no closed-loop equilibria in $\mathcal{G}$. Because $Ax + Bu + a \in \mathcal{B}$ for all $x \in \mathcal{O}$, $u \in \mathbb{R}^m$, it suffices to study existence of maps $F : \mathcal{G} \to \mathcal{B}$ without specific reference to the system (2.2). Further, using a straightforward linear algebra argument, one can show that either $\mathcal{O} = \emptyset$ or $\mathcal{O}$ is an affine space with $m \leq \dim(\mathcal{O}) \leq n$. Throughout the paper we assume $\mathcal{G} \neq \emptyset$. We can now state the main problem of a topological obstruction to solving the RCP by continuous state feedback.

Problem 2. *Given $\mathcal{S}$ and $\mathcal{O}$, find checkable conditions on $\mathcal{B}$ such that there exists a boundary map $f : \partial\mathcal{G} \to \mathcal{B}$ satisfying* (2.3) *that is extendible to a non-vanishing continuous map $F : \mathcal{G} \to \mathcal{B}$.*

As discussed above, Problem 2 is a necessary condition for the solvability of Problem 1. More precisely, if there exists a feedback $u(x)$ satisfying Problem 1, then the restriction of $Ax+Bu(x)+a$ to $\partial\mathcal{G}$ will be extendible to a non-vanishing map $F : \mathcal{G} \to \mathcal{B}$, again defined by $F(x) = Ax + Bu(x) + a$.

The central result of this paper is Theorem 21, which solves Problem 2 under certain assumptions on the geometry of $\mathcal{G}$ and $\mathcal{B}$. In order to preserve the narrative of this paper, these assumptions, denoted by (A2') and (A3), are formally presented and discussed at a later point in the paper. However, we provide a short description of them at this point.

Assumption (A2') concerns the geometry of the set $\mathcal{B}$. It states that, if $o_1, \ldots, o_{\kappa+1}$, $\kappa \geq m$, are vertices of $\mathcal{G}$, then there exists an $r \in \{2, \ldots, m+1\}$ such that $\mathcal{B}$ has a basis of vectors $b_1, \ldots, b_{r-1}, b_{r+1}, \ldots, b_{m+1}$ with $b_i \in \mathcal{C}(o_i)$ and $\mathcal{B} \cap \mathcal{C}(o_r) \subset \operatorname{span}\{b_1, \ldots, b_{r-1}\}$.

Assumption (A3) concerns the geometry of the set $\mathcal{G}$. It imposes a certain symmetry on $\mathcal{G}$, by stating that, if $o_1, \ldots, o_{\kappa+1}$ are vertices of $\mathcal{G}$, and $I(o_j)$ is the smallest set of vertex indices such that $o_j \in \operatorname{co}\{v_i \mid i \in I(o_j)\}$, then each $I(o_j)$ has one index which does not appear in any other $I(o_j)$, and all indices that are shared between at least two $I(o_j)$'s are shared between all.

The following theorem provides a necessary condition for the solvability of Problem 1. It is derived directly from the foregoing discussion and the claim of Theorem 21, which will be proved at the end of this paper.

Theorem 3. *Suppose there exists κ, $m \leq \kappa \leq n-1$, such that $\mathcal{G}$ is a κ-dimensional simplex. Additionally, suppose assumptions (A2') and (A3) hold. If there exists a continuous feedback $u(x)$ which solves Problem 1, then*

$$\mathcal{B} \cap \operatorname{cone}(\mathcal{G}) \neq \mathbf{0}\,.$$

We remarked in the introduction that the RCP was recently [24] posed as a problem in positive systems. In particular, if $\mathcal{S}$ is the "corner" of the positive orthant (i.e., $v_0 = 0$ and v_i, $i \in \{1, \ldots, n\}$ are the unit vectors on coordinate axes), then the above invariance conditions are equivalent to an affine system with a given feedback control is positive. Thus, Theorem 3 can be posed as a necessary condition for a positive system to lack equilibria in $\mathcal{S}$. As this paper primarily focuses on the RCP, we do not explore this connection further.

3. Mathematical Background

This paper employs an interplay of results from degree theory regarding maps in spheres, combinatorial results centered on the KKM lemma, and results from linear algebra on $\mathscr{M}$-matrices. This section gives the required background on degree theory and the KKM lemma. Degree theory is used for the proof (in the Appendix) of the main result of Section 4 on a topological obstruction on the sphere. The KKM lemma is used for the proof of the main result of Section 5.

3.1. Degree Theory

The degree of a continuous map is a multi-dimensional analogue of a winding number; that is, it gives the number of times a continuous map wraps around a manifold. We forego a rigorous definition since it involves homology theory and since we only use the properties of the degree; see [15] for an excellent overview. The following facts about degree are well-known.

Lemma 4.

(i) The degree of $id_{\mathbb{S}^n}$ is 1.

(ii) The degree of a constant map (i.e., $f : \mathbb{S}^n \to \mathbb{S}^n$ defined by $f(x) = x_0$ for all $x \in \mathbb{S}^n$) is 0.

(iii) Let $f : \mathbb{S}^n \to \mathbb{S}^n$ be the antipodal map $f(x) = -x$. Then $\deg(f) = (-1)^{n+1}$.

The notion of homotopy regards when one map can be continuously deformed into another and is closely related to degree theory.

Definition 5. *Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces, and let $f, g : \mathcal{X} \to \mathcal{Y}$ be continuous functions. Functions f and g are* homotopic *if there exists a continuous function $H : \mathcal{X} \times [0,1] \to \mathcal{Y}$ such that $H(\cdot, 0) \equiv f$ and $H(\cdot, 1) \equiv g$.*

Analogously to the notion of homotopy of maps, we can investigate when spaces can be continuously deformed into one another. In this paper, we forgo more complicated concepts and merely remind the reader of the notion of a homeomorphism:

Definition 6. *Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces. If there exists a continuous function $f : \mathcal{X} \to \mathcal{Y}$ which contains a continuous inverse, then $\mathcal{X}$ and $\mathcal{Y}$ are* homeomorphic *(denoted by $\mathcal{X} \sim \mathcal{Y}$).*

In the sense of topology, homeomorphic spaces are understood to be essentially the same, and all topological properties of a space are invariant to homeomorphisms.

Lemma 7. *Continuous functions $f : \mathbb{S}^n \to \mathbb{S}^n$ and $g : \mathbb{S}^n \to \mathbb{S}^n$ are homotopic if and only if $\deg(f) = \deg(g)$.*

Corollary 8. *A continuous function $f : \mathbb{S}^n \to \mathbb{S}^n$ is homotopic to a constant map if and only if $\deg(f) = 0$.*

The following result will be of particular interest to us. The proof is not difficult, and is a special case of the known result for functions on a sphere with no fixed points (for more detail, see, e.g., [15]).

Lemma 9. *Suppose a continuous function $f : \mathbb{S}^n \to \mathbb{S}^n$ satisfies $x \cdot f(x) \leq 0$, $x \in \mathbb{S}^n$. Then $\deg(f) = (-1)^{n+1}$.*

The central question of this paper regards one of the basic problems of algebraic topology, the *extension problem*: given a continuous map defined on the boundary of a topological space, when can it be continuously extended to a non-vanishing map in the interior of the set? The following well-known result answers this question, and is the main result of this section.

Theorem 10 (Extension Theorem)**.** *A continuous map $f : \mathbb{S}^n \to \mathbb{S}^n$ extends to a map $F : \mathbb{B}^{n+1} \to \mathbb{S}^n$ if and only if $\deg(f) = 0$.*

3.2. Knaster-Kuratowski-Mazurkiewicz Lemma

We use degree theory to study continuous extensions of maps defined on spheres. These results do not carry over naturally when we study simplices. As such, we draw upon the literature of fixed point theory and particularly the KKM lemma to resolve the question of extensions of functions on simplices. We employ the following variant of the KKM lemma.

Lemma 11 (Knaster-Kuratowski-Mazurkiewicz [4])**.** *Let $\mathcal{P} = \mathrm{co}\{w_1, \ldots, w_{n+1}\}$ be an n-dimensional simplex. Let $\{\mathcal{Q}_1, \ldots, \mathcal{Q}_{n+1}\}$ be a collection of sets covering $\mathcal{P}$. Consider the conditions:*

(P1) Vertex $w_i \in \mathcal{Q}_i$ and $w_i \notin \mathcal{Q}_j$ for $j \neq i$.

(P2) If $x \in \mathrm{co}\{w_{i_1}, \ldots, w_{i_l}\}$ for some $1 \leq l \leq n+1$, then $x \in \mathcal{Q}_{i_1} \cup \cdots \cup \mathcal{Q}_{i_l}$.

If (P1)-(P2) hold, then $\bigcap_{1=1}^{n+1} \overline{\mathcal{Q}}_i \neq \emptyset$.

An illustration of Lemma 11 is provided in Figure 2.

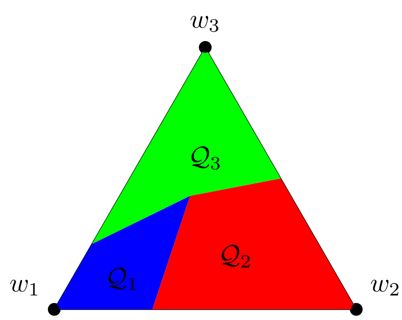

Figure 2: An illustration of Lemma 11. We note that $\overline{\mathcal{Q}}_1$, $\overline{\mathcal{Q}}_2$ and $\overline{\mathcal{Q}}_3$ intersect at a single point.

4. Topological Obstruction on the Sphere

Our search for checkable conditions on $\mathcal{B}$ to solve Problem 2 begins with an exploration of the analogous problem posed on the sphere so that standard tools of algebraic topology, particularly Theorem 10, can be brought to bear. Thus, in this section rather than working on the simplex $\mathcal{S}$, we consider instead $\mathbb{B}^n$, the closed unit ball in $\mathbb{R}^n$. Its boundary is $\mathbb{S}^{n-1} = \partial\mathbb{B}^n$. Other elements of Problem 2 remain the same. Let $\mathcal{B} \subset \mathbb{R}^n$ be a subspace of dimension $1 \leq m < n$, and let $\mathcal{O}$ be a subspace of $\dim(\mathcal{O}) = \kappa$ with $m \leq \kappa < n$. Define the set

$$\mathcal{G} = \mathcal{O} \cap \mathbb{B}^n$$

and define the boundary spheres $\partial\mathcal{G} := \mathcal{O} \cap \mathbb{S}^{n-1} \sim \mathbb{S}^{\kappa-1}$ and $\mathcal{B}_1 := \mathcal{B} \cap \mathbb{S}^{n-1} \sim \mathbb{S}^{m-1}$. Consider a continuous map $F : \mathcal{G} \to \mathcal{B}$ and define the boundary map $f = \partial F \ : \ \partial\mathcal{G} \to \mathcal{B}$. Suppose f satisfies

$$f(x) \neq 0\,, \qquad x \in \partial\mathcal{G} \tag{4.1}$$

$$x \cdot f(x) \leq 0\,, \qquad x \in \partial\mathcal{G}\,. \tag{4.2}$$

The second condition means $f(x)$ points inside the unit ball along its boundary, in analogy with the invariance conditions (2.3) for the simplex.

Example 12. *Figure 3 illustrates the situation for $n = 3$, $\kappa = 2$, and $m = 2$. The affine space $\mathcal{O}$ is a horizontal plane cutting the sphere $\mathbb{S}^2$ along the circle $\partial\mathcal{G}$. The set $\mathcal{G} = \mathcal{O} \cap \mathbb{B}^3$ is a closed ball. Condition* (4.2) *is illustrated for representative points along $\partial\mathcal{G}$. The subspace $\mathcal{B}$ is shown attached to several points in $\partial\mathcal{G}$. The (blue) vectors attached at $x \in \partial\mathcal{G}$ depict $f(x)$. They lie in the subspace $\mathcal{B}$ as well as point inside the sphere $\mathbb{S}^2$.*

In order to invoke results from degree theory, we define the map

$$y(x) := \frac{f(x)}{\|f(x)\|}\,, \qquad x \in \partial\mathcal{G}\,.$$

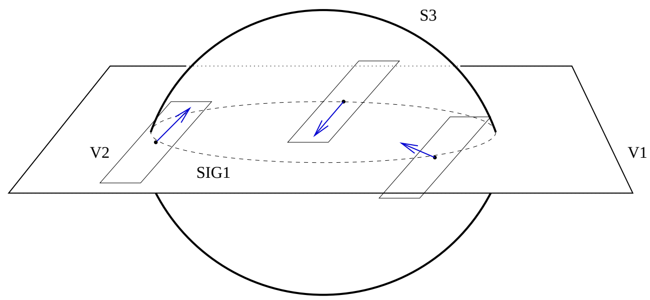

Figure 3: Graphic for Problem 13.

If $f(x) \neq 0$ on $\partial\mathcal{G}$, this map is well-defined. Observe that $y : \partial\mathcal{G} \to \mathcal{B}_1$ is a map on spheres and by (4.2),

$$x \cdot y(x) \leq 0\,, \qquad x \in \partial\mathcal{G}\,. \tag{4.3}$$

We want to know when y can be extended to a continuous map $Y : \mathcal{G} \to \mathcal{B}_1$. By Theorem 10 this is possible if and only if $\deg(y) = 0$. Suppose that F is non-vanishing. Then we can define an extension of F by $Y(x) = \frac{F(x)}{\|F(x)\|}$. Therefore, if y cannot be extended, the defect must be that there exists $\overline{x} \in \mathcal{G}$ such that $F(\overline{x}) = 0$. In other words, there exists a *topological obstruction* to finding a non-vanishing continuous map satisfying the inward pointing condition (4.2).

Problem 13. *Given $\mathcal{O}$, find conditions on $\mathcal{B}$ such that there exists a continuous map $y : \partial\mathcal{G} \to \mathcal{B}_1$ satisfying* (4.3) *that is continuously extendible to a non-vanishing map $Y : \mathcal{G} \to \mathcal{B}_1$.*

Remark 14. *While Problem 2 and Problem 13 are similar in flavour, there is no direct connection between the solutions to the two problems. Problem 2 forms a necessary condition for the solvability of the RCP. On the other hand, the role of Problem 13 is, in the context of the problem dealt with in this paper, purely motivational.*

Now we give our main idea to solve this problem. We know according to Theorem 10 that for y to be extendible, it must be homotopic to a constant map. Informally, it seems reasonable then to impose conditions on $\mathcal{B}$ such that a constant map can satisfy the inward pointing conditions (4.3) simultaneously at every point on the sphere $\partial\mathcal{G}$. In other words, the answer is positive if there exists a direction in $\mathcal{B}$ that points inside $\mathbb{B}^n$ at every point in $\mathcal{G}$. That is,

$$\mathcal{B} \cap \bigcap_{x \in \mathcal{G}} T_x\mathbb{B}^n \neq \mathbf{0}\,,$$

where $T_x\mathbb{B}^n$ is the Bouligand tangent cone to $\mathbb{B}^n$ at x [8]. Define the cone

$$\operatorname{cone}(\mathcal{G}) := \bigcap_{x\in\mathcal{G}} T_x\mathbb{B}^n = \left\{y \in \mathbb{R}^n \mid x \cdot y \le 0\,, x \in \partial\mathcal{G}\right\}. \tag{4.4}$$

More formally, the next result shows that indeed the proposed condition provides the solution to Problem 13. The proof is provided in the Appendix.

Theorem 15. *Consider subspace $\mathcal{B}$, affine space $\mathcal{O}$, and set $\mathcal{G} = \mathcal{O} \cap \mathbb{B}^n$. There exists a continuous map $y : \partial\mathcal{G} \to \mathcal{B}_1$ satisfying* (4.3) *that can be continuously extended to a map $Y : \mathcal{G} \to \mathcal{B}_1$ if and only if*

$$\mathcal{B} \cap \operatorname{cone}(\mathcal{G}) \neq \mathbf{0}\,. \tag{4.5}$$

5. A Topological Obstruction on the Simplex

In this section we exploit the main idea of the previous section to resolve the topological obstruction on the simplex. Rather than using a degree theory argument as for the sphere, we use a combinatorial argument often used in mathematical economics [4] which is generally based on the KKM lemma, Sperner's lemma, Scarf's lemma, or some variant of these.

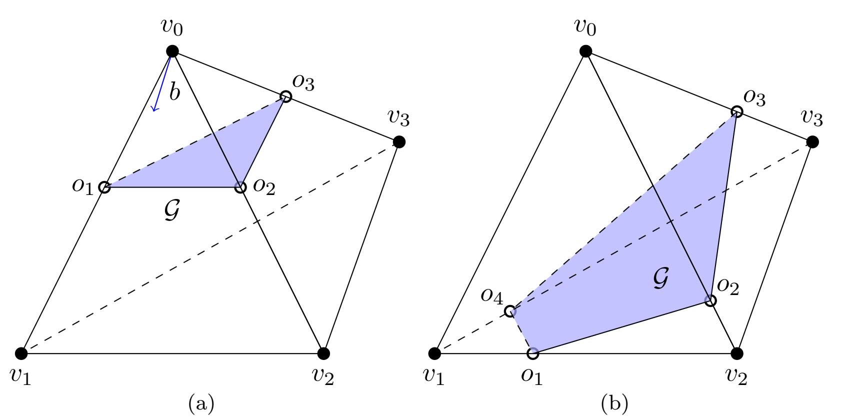

Figure 4: (a) $\mathcal{G}$ satisfies (A1), and $\mathcal{B} \cap \operatorname{cone}(\mathcal{G}) = \mathcal{B} \cap \mathcal{C}(v_0)$ contains $b \neq \mathbf{0}$. (b) $\mathcal{G}$ violates (A1).

As in Section 2, we consider an n-dimensional simplex $\mathcal{S} = \operatorname{co}\{v_0, \dots, v_n\}$ with vertex set $V := \{v_0, \dots, v_n\}$, facets $\mathcal{F}_0, \dots, \mathcal{F}_n$, and outward normal vectors $h_0, \dots, h_n$. Let $\mathcal{C}(x)$, $x \in \mathcal{S}$, be as in (2.1). Let $\mathcal{B}$, $\mathcal{O}$, and $\mathcal{G}$ be as in Section 2. We will additionally be making use of the following two assumptions:

(A1) There exists κ with $m \le \kappa \le n-1$ such that $\mathcal{G}$ is a κ-dimensional simplex.

(A2) There exist $b_1, \dots, b_i \in \mathbb{R}^n$ such that $b_i \in \mathcal{C}(o_i)$ for all $i \in \{1, \dots, m\}$ and $\mathcal{B} = \operatorname{span}\{b_1, \dots, b_m \mid b_i \in \mathcal{C}(o_i)\}$.

If assumption (A1) is satisfied, the vertices of $\mathcal{G}$ will be denoted by $o_1, \ldots, o_{\kappa+1}$.

The geometry that arises from the intersection of the affine space $\mathcal{O}$ with the simplex $\mathcal{S}$ is considerably more complex than the intersection of an affine space with a ball, which always results in a ball. Condition (A1) allows us to contain the geometric complexity of $\mathcal{G}$, and it can be guaranteed by a correct triangulation of the constrained state space. An illustration of (A1) is provided in Figure 4, modified from [20, 22].

Let the set of vertices of $\mathcal{G}$ be $V_{\mathcal{G}} := \{o_1, \ldots, o_{\kappa+1}\}$. Condition (A2) says that $\mathcal{B}$ does not have sufficiently high dimension to allow an assignment of linearly independent vectors at the vertices of $\mathcal{G}$. The motivation for this assumption is that if $\mathcal{G}$ is a simplex and there exists a linearly independent set $\{b_1, \ldots, b_{\kappa+1} \mid b_i \in \mathcal{C}(o_i)\}$, then trivially, $0 \notin \mathrm{co}\{b_1, \ldots, b_{\kappa+1}\}$, so one can construct an affine feedback $u(x)$ so there are no equilibria on $\mathcal{G}$. In essence, when the system is sufficiently actuated, the topological obstruction does not arise.

Define the cone

$$\mathrm{cone}(\mathcal{G}) := \bigcap_{x \in \mathcal{G}} \mathcal{C}(x) \,. \tag{5.1}$$

This definition is inspired by our studies on the sphere, but due to the properties of the simplex and our index convention, a more explicit characterization is now available.

$$\begin{aligned} \mathrm{cone}(\mathcal{G}) = \bigcap_{i=1}^{\kappa+1} \mathcal{C}(o_i) &= \bigcap_{i=1}^{\kappa+1} \Big\{ y \mid h_j \cdot y \leq 0 \,, \ \ j \in I \setminus I(o_i) \Big\} \\ &= \Big\{ y \mid h_j \cdot y \leq 0 \,, \ j \in I \setminus [I(o_1) \cap \cdots \cap I(o_{\kappa+1})] \Big\} \end{aligned} \tag{5.2}$$

Now consider a vertex map $f : V_{\mathcal{G}} \to \mathcal{B}$ and suppose that

$$f(o_i) \in \mathcal{C}(o_i) \,, \quad i = 1, \ldots, \kappa + 1 \,. \tag{5.3}$$

Given $\mathcal{S}$ and $\mathcal{O}$, we want to find conditions on $\mathcal{B}$ such that there exists a vertex map $f : V_{\mathcal{G}} \to \mathcal{B}$ satisfying (5.3) that is extendible to a non-vanishing continuous map $F : \mathcal{G} \to \mathcal{B}$ satisfying

$$F(x) \in \mathcal{C}(x) \,, \quad x \in \mathcal{G} \,. \tag{5.4}$$

Clearly, this problem is equivalent to Problem 2: if there exists a vertex map $f : V_{\mathcal{G}} \to \mathcal{B}$ that is extendible to $F : \mathcal{G} \to \mathcal{B}$, then the map $\partial F : \partial\mathcal{G} \to \mathcal{B}$ given by the restriction of F to $\partial\mathcal{G}$ is also extendible to $F : \mathcal{G} \to \mathcal{B}$.

Consider the cone $\mathcal{B} \cap \mathcal{C}(o_{m+1})$. Clearly, $\mathcal{B} \cap \mathcal{C}(o_{m+1}) \subset \mathrm{span}\{b_1, \ldots, b_m\}$. Moreover, assuming $\mathcal{B} \cap \mathcal{C}(o_{m+1}) \neq \mathbf{0}$, there exists an integer $2 \leq r \leq m+1$ such that without loss of generality (by reordering indices $1, \ldots, m$), $\mathcal{B} \cap \mathcal{C}(o_{m+1}) \subset \mathrm{span}\{b_1, \ldots, b_{r-1}\}$ and $\mathrm{span}\{b_1, \ldots, b_{r-1}\}$ is the smallest subspace generated by basis vectors among $\{b_1, \ldots, b_m\}$ that contains the cone $\mathcal{B} \cap \mathcal{C}(o_{m+1})$. Indeed it can be shown that such a minimal subspace is unique [7]. In order to have consecutive indices, it is useful to renumber the vertices of $\mathcal{G}$ to effectively swap the indices $m + 1$ and r, so we get

$$\mathcal{B} \cap \mathcal{C}(o_r) \subset \text{span}\{b_1, \ldots, b_{r-1}\}\,. \tag{5.5}$$

Because of the index swap between r and $m + 1$, a basis for $\mathcal{B}$ is

$$\mathcal{B} = \text{span}\{b_1, \ldots, b_{r-1}, b_{r+1}, \ldots, b_{m+1} \mid b_i \in \mathcal{C}(o_i)\}\,. \tag{5.6}$$

The restatement of Assumption (A2) along the lines of the previous paragraph, i.e., such that (5.5) and (5.6) hold, will be denoted as assumption (A2'). We emphasize that this is essentially the same assumption as (A2), with a mere reordering of vertices for easier notation.

The next result says that there exists a vector in the cone $\mathcal{B} \cap \mathcal{C}(o_r)$ that depends on all the vectors $\{b_1, \ldots, b_{r-1}\}$.

Lemma 16 ([7])**.** *Suppose (A1) and (A2') hold. There exists* $\overline{b}_r \in \mathcal{B} \cap \mathcal{C}(o_r)$ *such that*

$$\overline{b}_r = \overline{c}_1 b_1 + \cdots + \overline{c}_{r-1} b_{r-1}\,, \qquad \overline{c}_i \neq 0, \quad i = 1, \ldots, r-1\,. \tag{5.7}$$

The arguments to follow will involve manipulations with the index sets $I(o_i)$. To that end, for each $i = 1, \ldots, \kappa + 1$ define $E(o_i) \subset I(o_i)$ to be the set of non-zero *exclusive members* of $I(o_i)$ given by

$$E(o_i) := \big\{k \in I(o_i) \mid k \notin I(o_j), \forall j \neq i\big\} \setminus \{0\}\,.$$

We also define S to be the set of non-zero *shared vertices* given by

$$S = \Big[\bigcup_{i=1}^{\kappa+1} I(o_i)\Big] \;\setminus\; \Big[\bigcup_{i=1}^{\kappa+1} E(o_i)\Big]\,.$$

Observe that $S \cap E(o_i) = \emptyset$ for $i \in \{1, \ldots, \kappa + 1\}$ and $E(o_i) \cap E(o_j) = \emptyset$ for $i \neq j$.

Lemma 17. *Suppose (A1) and (A2') hold. Let* $\overline{b}_r \in \mathcal{B} \cap \mathcal{C}(o_r)$ *be given by* (5.7). *If* $\mathcal{B} \cap \{y \in \mathbb{R}^n \mid h_j \cdot y \leq 0, j \in I \setminus S\} = \mathbf{0}$, *then* $\overline{c}_i < 0$, $i = 1, \ldots, r-1$.

Proof. For ease of notation, let $b_r := \overline{b}_r$. First we show $0 \in \text{co}\{b_1, \ldots, b_r\}$. Since by (5.7), $\{b_1, \ldots, b_r\}$ are linearly dependent, there exist $\alpha_i \neq 0$ such that

$$\alpha_1 b_1 + \cdots + \alpha_r b_r = 0\,. \tag{5.8}$$

If all $\alpha_i < 0$ or all $\alpha_i > 0$, then it easily follows that $0 \in \text{co}\{b_1, \ldots, b_r\}$, as desired. Otherwise there must be $1 \leq q \leq r - 1$ such that without loss of generality $\alpha_1, \ldots, \alpha_q < 0$ and $\alpha_{q+1}, \ldots, \alpha_r > 0$. Define

$$\eta = -\alpha_1 b_1 - \cdots - \alpha_q b_q = \alpha_{q+1} b_{q+1} + \cdots + \alpha_r b_r\,.$$

Since $b_i \in \mathcal{B} \cap \mathcal{C}(o_i)$, $i = 1, \ldots, \kappa + 1$, we have

$$
\begin{aligned}
h_j \cdot \eta &= h_j \cdot (-\alpha_1 b_1 - \cdots - \alpha_q b_q) \le 0\,, \quad j = I \setminus I(o_1) \cap \cdots \cap I \setminus I(o_q) \\
h_j \cdot \eta &= h_j \cdot (\alpha_{q+1} b_{q+1} + \cdots + \alpha_r b_r) \le 0\,, \quad j = I \setminus I(o_{q+1}) \cap \cdots \cap I \setminus I(o_r)\,.
\end{aligned}
$$

Now we observe $I(o_1) \cup \cdots \cup I(o_q) \subset S \cup E(o_1) \cup \cdots \cup E(o_q)$. Thus $I \setminus [S \cup E(o_1) \cup \cdots \cup E(o_q)] \subset I \setminus I(o_1) \cap \cdots \cap I \setminus I(o_q)$. Similarly, $I \setminus [S \cup E(o_{q+1}) \cup \cdots \cup E(o_r)] \subset I \setminus I(o_{q+1}) \cap \cdots \cap I \setminus I(o_r)$. Also because $[E(o_1) \cup \cdots \cup E(o_q)] \cap [E(o_{q+1}) \cup \cdots \cup E(o_r)] = \emptyset$,

$$
\begin{aligned}
& \Big(I \setminus (S \cup E(o_1) \cup \cdots \cup E(o_q))\Big) \bigcup \Big(I \setminus (S \cup E(o_{q+1}) \cup \cdots \cup E(o_r))\Big) \\
= \;& I \setminus \Big[(S \cup E(o_1) \cup \cdots \cup E(o_q)) \cap (S \cup E(o_{q+1}) \cup \cdots \cup E(o_r))\Big] = I \setminus S\,.
\end{aligned}
$$

Thus, $I \setminus S \subset [I \setminus I(o_1) \cap \cdots \cap I \setminus I(o_q)] \cup [I \setminus I(o_{q+1}) \cap \cdots \cap I \setminus I(o_r)]$. We conclude $\eta \in \mathcal{B} \cap \big\{y \mid h_j \cdot y \le 0\,,\ \ j \in I \setminus S\big\}$, so $\eta = 0$. Then

$$
0 = \frac{\alpha_{q+1} b_{q+1} + \cdots + \alpha_r b_r}{\alpha_{q+1} + \cdots + \alpha_r} \in \mathrm{co}\{b_1, \ldots, b_r\}\,,
$$

as desired. As a consequence, we can assume without loss of generality that $\alpha_i > 0$ in (5.8). Hence,

$$
\bar{b}_r = -\frac{\alpha_1}{\alpha_r} b_1 - \cdots - \frac{\alpha_{r-1}}{\alpha_r} b_{r-1}\,. \tag{5.9}
$$

Comparing (5.9) with (5.7) and again using the fact that $\{b_1, \ldots, b_{r-1}\}$ are linearly independent, we get $\bar{c}_i = -\frac{\alpha_i}{\alpha_r} < 0$, $i = 1, \ldots, r-1$, as desired. ☐

Lemma 18. *Suppose (A1) and (A2') hold. If* $\mathcal{B} \cap \{y \in \mathbb{R}^n \mid h_j \cdot y \le 0, j \in I \setminus S\} = \mathbf{0}$*, then for any* $\beta_r \in \mathcal{B} \cap \mathcal{C}(o_r)$*,*

$$
\begin{aligned}
h_j \cdot b_i &= 0\,, \quad i = 1, \ldots, r-1\,, \quad j \in [I \setminus I(o_1)] \cap \cdots \cap [I \setminus I(o_r)] \\
h_j \cdot \beta_r &= 0\,, \quad j \in [I \setminus I(o_1)] \cap \cdots \cap [I \setminus I(o_r)]\,.
\end{aligned}
$$

Proof. Using (5.7) we have

$$
h_j \cdot \big(\bar{b}_r - \bar{c}_1 b_1 - \cdots - \bar{c}_{r-1} b_{r-1}\big) = 0\,, \qquad j \in I \setminus [I(o_1) \cup \cdots \cup I(o_r)]\,.
$$

Expanding the product on the left side, every term $-\bar{c}_i h_j \cdot b_i$ in the sum is non-positive because $b_i \in \mathcal{B} \cap \mathcal{C}(o_i)$ and $-c_i > 0$ by Lemma 17. Additionally, $h_j \cdot \bar{b}_r \le 0$ because $\bar{b}_r \in \mathcal{B} \cap \mathcal{C}(o_r)$. Thus,

$$
h_j \cdot b_i = 0\,, \qquad i = 1, \ldots, r-1\,, \quad j \in I \setminus [I(o_1) \cup \cdots \cup I(o_r)]\,. \tag{5.10}
$$

Now take any $\beta_r \in \mathcal{B} \cap \mathcal{C}(o_r)$. By (5.5) and (5.10) we have

$$h_j \cdot \beta_r = 0\,, \qquad j \in I \setminus [I(o_1) \cup \cdots \cup I(o_r)]\,.$$

$\square$

In order to mimic as closely as possible the topological obstruction on the sphere, we now introduce an assumption on the geometry of the intersection of $\mathcal{O}$ with $\mathcal{S}$. It is captured in terms of the index sets $I(o_i)$. Let $E(o_i)$ be the set of non-zero exclusive vertices of o_i for $i = 1, \ldots, \kappa+1$, and let S be the set of non-zero shared vertices. We assume the following:

(A3) For each $i \in \{1, \ldots, \kappa+1\}$, there exists $e_i \in \{1, \ldots, n\}$ such that $E(o_i) = \{e_i\}$. Moreover, $S = [I(o_1) \cap \cdots \cap I(o_{\kappa+1})] \setminus \{0\}$.

This assumption on the index sets has several implications which we now spell out. First

$$[I \setminus I(o_1)] \cup \cdots \cup [I \setminus I(o_{\kappa+1})] = I \setminus [I(o_1) \cap \cdots \cap I(o_{\kappa+1})] = I \setminus S\,. \tag{5.11}$$

Therefore, under (A3)

$$\mathrm{cone}(\mathcal{G}) = \Big\{ y \in \mathbb{R}^n \mid h_j \cdot y \le 0\,, \;\; j \in I \setminus S \Big\}\,. \tag{5.12}$$

Second, for all $1 \le p \le r$ we have

$$\big([I \setminus I(o_1)] \cap \cdots \cap [I \setminus I(o_p)]\big) \bigcup \{e_1, \ldots, e_p\} = I \setminus S\,. \tag{5.13}$$

Our main result relies on properties of $\mathscr{Z}$-matrices and $\mathscr{M}$-matrices. We say a matrix M is a $\mathscr{Z}$-matrix if the off-diagonal elements are non-positive; i.e. $m_{ij} \le 0$ for all $i \ne j$. A $\mathscr{Z}$-matrix M is a nonsingular $\mathscr{M}$-matrix if every real eigenvalue of M is positive [3]. Define the matrices

$$H := \big[\, h_{e_1} \cdots \, h_{e_{r-1}} \big]\,, \quad Y := \big[\, b_1 \; \cdots \; b_{r-1} \big]\,, \quad M := H^T Y\,. \tag{5.14}$$

Lemma 19. *Suppose (A1), (A2'), and (A3) hold. Also suppose* $\mathcal{B} \cap \mathrm{cone}(\mathcal{G}) = \mathbf{0}$*. Then* M *is a nonsingular* $\mathscr{M}$*-matrix.*

Proof. First, we show M is a $\mathscr{Z}$-matrix. Observe that since $\{e_1, \ldots, e_{r-1}\} \setminus \{e_i\} \subset I \setminus I(o_i)$,

$$h_j \cdot b_i \le 0\,, \qquad j \in \{e_1, \ldots, e_{r-1}\} \setminus \{e_i\}\,. \tag{5.15}$$

The inequalities (5.15) imply M is a $\mathscr{Z}$-matrix.

Next we show that M is nonsingular. Suppose there exists $c \in \mathbb{R}^{r-1}$ such that $H^T Y c = 0$. Let $y := Yc$. Then $h_j \cdot y = 0$, $j = 1, \ldots, r-1$. Also by Lemma 18, $h_{e_j} \cdot y = 0$, $j \in [I \setminus I(o_1)] \cap \cdots \cap [I \setminus I(o_r)]$. However, $[I \setminus (o_1)] \cap \cdots \cap [I \setminus I(o_r)] = I \setminus [S \cup \{e_1, \ldots, e_r\}]$. We conclude that $h_j \cdot y = 0$, $j \in I \setminus [S \cup \{e_r\}]$. Now if $h_{e_r} \cdot y \le 0$, then $h_j \cdot y \le 0$, $j \in I \setminus S$. Then by (A3) and (5.12), we get

$y \in \mathcal{B} \cap \text{cone}(\mathcal{G})$; otherwise if $h_{e_r} \cdot y > 0$ we get $-y \in \mathcal{B} \cap \text{cone}(\mathcal{G})$. Then by assumption, $y = 0$. However, $y = c_1 b_1 + \cdots + c_{r-1} b_{r-1}$ and $\{b_1, \ldots, b_{r-1}\}$ are linearly independent, so $c = 0$. We conclude that M is nonsingular.

Finally, we show M satisfies case (Q_{50}) of Theorem 6.2.3 of [3]. Suppose there exists $c \in \mathbb{R}^{r-1}$ with $c \neq 0$ and $c \succeq 0$ such that $Mc \preceq 0$, where $\succeq$ and $\preceq$ denote component-wise relations: i.e., every coordinate component of c is greater than or equal to 0 (analogously, less than or equal to 0). Define the vector $y := Yc \in \mathcal{B}$. Note that $y \neq 0$ because $\{b_1, \ldots, b_{r-1}\}$ are linearly independent. Then $Mc = H^T Y c = H^T y \preceq 0$ implies $h_{e_j} \cdot y \leq 0$, $j = 1, \ldots, r-1$. Also, since $c_i \geq 0$ and $b_i \in \mathcal{B} \cap \mathcal{C}(o_i)$,

$$h_j \cdot y = \sum_{i=1}^{r-1} c_i (h_j \cdot b_i) \leq 0, \qquad j \in I \setminus I(o_1) \cap \cdots \cap I \setminus I(o_{r-1}) \,.$$

Combining the previous two inequalities and using (5.13), we get $h_j \cdot y \leq 0$, $j \in I \setminus S$. This implies $0 \neq y \in \mathcal{B} \cap \text{cone}(\mathcal{G})$, a contradiction. Therefore, M has the property that the only solution of the inequalities $c \succeq 0$ and $Mc \preceq 0$ is $c = 0$. In sum, M is a nonsingular $\mathscr{Z}$-matrix satisfying Theorem 6.2.3, case (Q_{50}) of [3], so M is a nonsingular $\mathscr{M}$-matrix. □

Now let $F : \mathcal{G} \to \mathcal{B}$ be a map. In light of (A2'),

$$F(x) = c_1(x) b_1 + \cdots + c_{r-1}(x) b_{r-1} + \beta(x) \,, \qquad x \in \mathcal{G} \tag{5.16}$$

where $c_i : \mathcal{G} \to \mathbb{R}$ for $i = 1, \ldots, r-1$, $\beta : \mathcal{G} \to \mathbb{R}$, and $\beta(x) \in \text{span}\{b_{r+1}, \ldots, b_{m+1}\}$. Since $\mathcal{G}$ is a simplex by (A1), we can define the simplex, a face of $\mathcal{G}$, as

$$\widehat{\mathcal{G}} := \text{co}\{o_1, \ldots, o_r\} \,. \tag{5.17}$$

Lemma 20. *Suppose (A1), (A2'), and (A3) hold and $\mathcal{B} \cap \text{cone}(\mathcal{G}) = \mathbf{0}$. Let $F : \mathcal{G} \to \mathcal{B}$ be a map of the form* (5.16) *and suppose F satisfies* (5.4). *Then $\beta(x) = 0$, $x \in \widehat{\mathcal{G}}$.*

Proof. By assumption, $F(x) \in \mathcal{C}(x)$, $x \in \mathcal{G}$. For all $x \in \widehat{\mathcal{G}}$, $I(x) \subset I(o_1) \cup \cdots \cup I(o_r)$. Thus, $h_j \cdot F(x) \leq 0$, $j \in I \setminus [I(o_1) \cup \cdots \cup I(o_r)]$. Using (5.16) and Lemma 18, for all $x \in \widehat{\mathcal{G}}$, $h_j \cdot F(x) = h_j \cdot \beta(x) \leq 0$, $j \in [I \setminus I(o_1)] \cap \cdots \cap [I \setminus I(o_r)]$. Now we argue that $\beta(x) = 0$, $x \in \widehat{\mathcal{G}}$. Suppose not. Then there exists $\overline{x} \in \widehat{\mathcal{G}}$ and $\overline{\beta} := \beta(\overline{x}) \neq 0$ such that

$$h_j \cdot \overline{\beta} \leq 0 \,, \quad j \in [I \setminus I(o_1)] \cap \cdots \cap [I \setminus I(o_r)] \,. \tag{5.18}$$

Consider $M = H^T Y$ where H and Y are defined in (5.14). By Lemma 19, M is a nonsingular $\mathscr{M}$-matrix. By Theorem 6.2.3, case (I_{28}) of [3], there exists $c' \preceq 0$ such that $Mc' \prec 0$. Define $b'_r := Yc'$. Then $Mc' = H^T b'_r \prec 0$; that is,

$$h_{e_j} \cdot b'_r < 0 \,, \qquad j = 1, \ldots, r-1 \,. \tag{5.19}$$

From Lemma 18,

$$h_j \cdot b'_r = h_j \cdot [c'_1 b_1 + \cdots + c'_{r-1} b_{r-1}] = 0\,, \qquad j \in [I \setminus I(o_1)] \cap \cdots \cap [I \setminus I(o_r)]\,. \tag{5.20}$$

However, analogous to the calculation for (5.13),

$$[I \setminus I(o_1)] \cap \cdots \cap [I \setminus I(o_r)] \bigcup \{e_1, \ldots, e_{r-1}\} = I \setminus (S \cup \{e_r\}) = I \setminus I(o_r).$$

Hence, $b'_r \in \mathcal{B} \cap \mathcal{C}(o_r)$. Now consider $b''_r := b'_r + \alpha\overline{\beta}$, where $\alpha > 0$ is a constant. Using (5.18)-(5.20) we can choose $\alpha > 0$ sufficiently small such that either

$$h_j \cdot b''_r = h_j \cdot (b'_r + \alpha\overline{\beta}) = h_j \cdot \overline{\beta} \leq 0, \qquad j \in [I \setminus I(o_1)] \cap \cdots \cap [I \setminus I(o_r)]\,,$$

or

$$h_{e_j} \cdot b''_r < 0\,, \qquad j = 1, \ldots, r-1\,.$$

Combining these two inequalities, $h_j \cdot b''_r \leq 0$, $j \in I \setminus I(o_r)$. That is, $b''_r \in \mathcal{B} \cap \mathcal{C}(o_r)$. Moreover, with $\overline{\beta} \neq 0$ and $\overline{\beta} \in \{ b_{r+1}, \ldots, b_{m+1}\}$, we have $b''_r \notin \{ b_1, \ldots, v_{r-1}\}$. Therefore, $\{b_1, \ldots, b_{r-1}, b''_r\}$ is a linearly independent set. This contradicts (5.5) that $\mathcal{B} \cap \mathcal{C}(o_r) \subset \mathrm{span}\{b_1, \ldots, b_{r-1}\}$. The conclusion is that there does not exist $x \in \widehat{\mathcal{G}}$ such that $\beta(x) \in \mathrm{span}\{b_{r+1}, \ldots, b_{m+1}\}$ and $\beta(x) \neq 0$. ☐

The following is the main result of the paper.

Theorem 21. *Suppose (A1), (A2'), and (A3) hold. There exists a vertex map $f : V_{\mathcal{G}} \to \mathcal{B}$ satisfying* (5.3) *that can be extended to a continuous, non-vanishing map $F : \mathcal{G} \to \mathcal{B}$ satisfying* (5.4) *if and only if*

$$\mathcal{B} \cap \mathrm{cone}(\mathcal{G}) \neq \mathbf{0}\,. \tag{5.21}$$

Proof. Suppose there exists a vertex map $f : V_{\mathcal{G}} \to \mathcal{B}$ satisfying (5.3) that can be extended to a continuous, non-vanishing map $F : \mathcal{G} \to \mathcal{B}$ satisfying (5.4), but $\mathcal{B} \cap \mathrm{cone}(\mathcal{G}) = \mathbf{0}$. Define $\widehat{\mathcal{G}}$ as in (5.17). By Lemma 20 we have that

$$F(x) = c_1(x) b_1 + \cdots c_{r-1}(x) b_{r-1}\,, \qquad x \in \widehat{\mathcal{G}}\,.$$

Because $F(x)$ is continuous and $\{b_1, \ldots, b_{r-1}\}$ are linearly independent, $c_i : \widehat{\mathcal{G}} \to \mathbb{R}$ are continuous functions. Define the sets

$$\mathcal{Q}_i := \left\{ x \in \widehat{\mathcal{G}} \mid h_{e_i} \cdot F(x) > 0 \right\}, \quad i = 1, \ldots, r\,. \tag{5.22}$$

Now we verify the conditions of Lemma 11.

First we claim that $\{\mathcal{Q}_i\}$ cover $\widehat{\mathcal{G}}$. For suppose not. Then there exists $x \in \widehat{\mathcal{G}}$ such that $h_{e_j} \cdot F(x) \leq 0$, $j = 1, \ldots, r$. By (5.4) we also have that $h_j \cdot F(x) \leq 0$, $j \in I \setminus I(x)$. Since $x \in \mathrm{co}\{o_1, \ldots, o_r\}$, $I(x) \subset I(o_1) \cup \cdots \cup I(o_r)$. Thus, $h_j \cdot F(x) \leq 0$, $j \in I \setminus [I(o_1) \cup \cdots \cup I(o_r)]$. Using (5.13) and (A3), this implies $F(x) \in \mathcal{B} \cap \mathrm{cone}(\mathcal{G})$, so $F(x) = 0$, a contradiction to F being non-vanishing on $\mathcal{G}$.

Second we verify property (P1). We claim that for each $i = 1, \ldots, r$, $o_i \in \mathcal{Q}_i$. For suppose not. Then by (5.4) and the assumption, $h_j \cdot F(x) \leq 0$, $j \in [I \setminus I(o_i)] \cup \{e_i\}$. But $[I \setminus I(o_i)] \cup \{e_i\} = I \setminus S$. Hence, $F(o_i) \in \mathcal{B} \cap \text{cone}(\mathcal{G})$, so $F(o_i) = 0$, a contradiction. Next we claim $o_i \notin \mathcal{Q}_k$, $k \neq i$. By (5.4), $h_j \cdot F(o_i) \leq 0$, $j \in I \setminus I(o_i)$ and by definition of $E(o_k)$, $e_k \in I \setminus I(o_i)$, $k \neq i$. Hence, $h_{e_k} \cdot F(o_i) \leq 0$, which means $o_i \notin \mathcal{Q}_k$.

Third we verify property (P2). Suppose without loss of generality (by re-ordering the indices $\{1, \ldots, r\}$) $x \in \text{co}\{o_1, \ldots, o_p\}$ for some $1 \leq p \leq r$. We claim $x \in \mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_p$. For suppose not. Then by (5.4) and the assumption $x \notin \mathcal{Q}_i$, $i = 1, \ldots, p$, $h_j \cdot F(x) \leq 0$, $j \in (I \setminus I(x)) \cup \{e_1, \ldots, e_p\}$. However, as we argued above, $I(x) \subset I(o_1) \cup \cdots \cup I(o_p)$. Then using (5.13), $I \setminus S \subset (I \setminus I(x)) \cup \{e_1, \ldots, e_p\}$. Hence, $F(x) \in \mathcal{B} \cap \text{cone}(\mathcal{G})$, so $F(x) = 0$, a contradiction to F being non-vanishing on $\mathcal{G}$.

We have verified (P1)-(P2) of Lemma 11. Applying the lemma, there exists $\overline{x} \in \bigcap_{i=1}^{r} \overline{\mathcal{Q}}_i$ such that $h_{e_i} \cdot F(\overline{x}) \geq 0$, $i = 1, \ldots, r$. Now by Lemma 20, $F(\overline{x}) = c_1(\overline{x})b_1 + \cdots + c_{r-1}(\overline{x})b_{r-1}$. Hence, by Lemma 18, $h_j \cdot F(\overline{x}) = 0$, $j \in I \setminus [I(o_1) \cup \cdots \cup I(o_r)]$. Using again (5.13) and (5.12), we conclude that $-F(\overline{x}) \in \mathcal{B} \cap \text{cone}(\mathcal{G})$, so $F(\overline{x}) = 0$, a contradiction. $\square$

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Appendix A.

Proof of Theorem 15. ($\Longleftarrow$) Suppose (4.5) holds. Pick $0 \neq z \in \mathcal{B} \cap \mathrm{cone}(\mathcal{G})$ and define $Y : \mathcal{G} \to \mathcal{B}_1$ as the constant map $Y(x) := \frac{z}{\|z\|}$. Clearly $Y(x)$ is continuous and by definition of $\mathrm{cone}(\mathcal{G})$, it satisfies (4.3).

($\Longrightarrow$) Suppose there exists a continuous map $y : \partial\mathcal{G} \to \mathcal{B}_1$ satisfying (4.3) that can be extended to a continuous map $Y : \mathcal{G} \to \mathcal{B}_1$ but suppose $\mathcal{B} \cap \mathrm{cone}(\mathcal{G}) = \mathbf{0}$. Without loss of generality, we can rotate the sphere as needed so that the affine space $\mathcal{O}$ is horizontal. See Figure A.5. Then we can choose coordinates so that the following definitions can be made:

$$
\begin{aligned}
p_0 &:= (0, \ldots, 0, \alpha)\,, \qquad\qquad \alpha \in (0,1] \\
p_1 &:= (0, \ldots, 0, \frac{1}{\alpha}) \\
\mathcal{V} &= \{x \in \mathbb{R}^n \mid x_{\kappa+1} = \cdots = x_n = 0\} = \mathbb{R}^\kappa \times \{0\}^{n-\kappa} \\
\mathcal{O} &= p_0 + \mathcal{V}\,.
\end{aligned}
$$

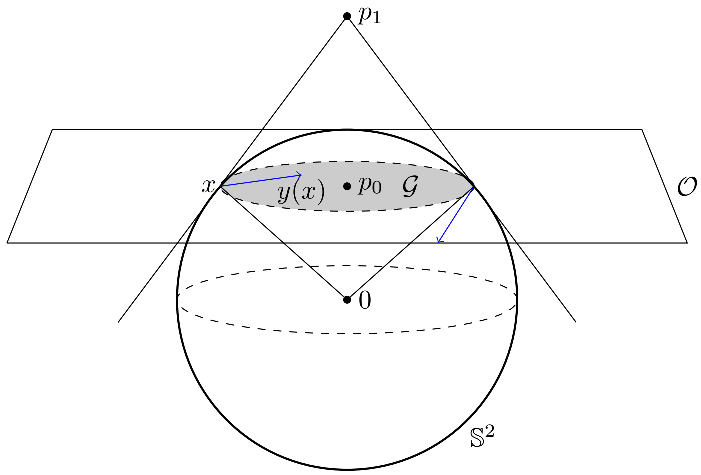

Figure A.5: Illustration for the proof of Theorem 15.

Point p_0 is the center of the ball $\mathcal{G}$ and p_1 has been selected to achieve a certain tangency property on the boundary of $\mathcal{G}$, as will be explained below. The subspace $\mathcal{V}$ is a parallel translate of $\mathcal{O}$. Additionally, we take a subspace orthogonal to $\mathcal{V}$

$$\mathcal{W} := \{x \in \mathbb{R}^n \mid x_1 = \cdots = x_\kappa = x_n = 0\} = \{0\}^\kappa \times \mathbb{R}^{n-\kappa-1} \times \{0\}\,,$$

so that $\mathcal{V} \oplus \mathcal{W} = \mathbb{R}^{n-1} \times \{0\}$. Now there are two cases. First, suppose $\alpha = 1$ so $p_0 = (0, \ldots, 0, 1)$. This means that $\mathcal{G} = \{p_0\}$ and $\text{cone}(\mathcal{G}) = \{y \mid p_0 \cdot y \le 0\}$, which is a closed-half space. However, by (4.3), $y(p_0) \in \mathcal{B} \cap \text{cone}(\mathcal{G}) = \mathbf{0}$. Thus, $y(p_0) = 0$. This is a contradiction since $y : \partial\mathcal{G} \to \mathcal{B}_1$ and $0 \notin \mathcal{B}_1$.

Second, suppose $\alpha < 1$. We note that for each $x \in \mathcal{G}$, the n-th coordinate of $x - p_1$ equals $\alpha - \frac{1}{\alpha} \neq 0$. Thus, for each $x \in \mathcal{G}$, $\mathbb{R}^n = (\mathbb{R}^{n-1} \times \{0\}) \oplus \text{span}\{x - p_1\}$. Thus, for each $x \in \mathcal{G}$,

$$\mathbb{R}^n = \mathcal{V} \oplus \text{span}\{x - p_1\} \oplus \mathcal{W}\,. \tag{A.1}$$

Let $x = (x_1, \ldots, x_\kappa, 0, \ldots, 0, \alpha) \in \mathcal{G}$. It can be verified by direct computation that $x \cdot (x - p_1) = \|x\|^2 - 1$, so

$$x \cdot (x - p_1) \begin{cases} = 0\,, & x \in \partial\mathcal{G} \\ \le 0\,, & x \in \mathcal{G}\,. \end{cases} \tag{A.2}$$

As shown in Figure A.5, the first part of (A.2) says that the ray from p_1 through any point x in the boundary of $\mathcal{G}$ is orthogonal to x. Now, let $v = (0, \ldots, 0, v_{\kappa+1}, \ldots, v_{n-1}, 0)$ be any point in $\mathcal{W}$. Again by direct computation

$$x \cdot v = 0\,, \qquad x \in \mathcal{G}\,, v \in \mathcal{W}\,. \tag{A.3}$$

Combining (A.2) and (A.3) we have

$$x \cdot y = 0\,, \qquad\qquad x \in \partial\mathcal{G}\,, y \in \mathrm{span}\{x - p_1\} \oplus \mathcal{W}\,. \tag{A.4}$$

Using (A.1) one obtains a unique decomposition of $Y(x)$ as

$$Y(x) = Y^t(x) + Y^r(x) + Y^c(x) \tag{A.5}$$

where $Y^t : \mathcal{G} \to \mathcal{V}$, $Y^r(x) \in \mathrm{span}\{x - p_1\}$, and $Y^c : \mathcal{G} \to \mathcal{W}$. The component Y^r has the form $Y^r(x) = \lambda(x)(x - p_1)$, where λ is a scalar function. Since Y^t and Y^c have their last coordinate equal to 0, then the last coordinate of Y satisfies $Y_n(x) = Y^r_n(x) = \lambda(x)(\frac{\alpha-1}{\alpha})$. Since Y is continuous, Y_n is also continuous, and so is $\lambda(x) = \frac{\alpha}{\alpha-1}Y_n(x)$. We conclude $Y^r(x) = \lambda(x)(x - p_1)$ and $Y - Y^r$ are continuous functions. Now $Y^t(x)$ and $Y^c(x)$ are obtained by projecting $Y - Y^r$ to $\mathcal{V}$ and $\mathcal{W}$, respectively (in fact, for $Y^t(x)$ we take the first κ coordinates of $Y(x) - Y^r(x)$, and for $Y^c(x)$ we take the $n - \kappa - 1$ coordinates after that). By an argument analogous to the one above, we can show both Y^t and Y^c are also continuous.

Now suppose $Y^t(x) \neq 0$ for all $x \in \mathcal{G}$. Let $\mathcal{G}^0 := \mathcal{V} \cap \mathbb{B}^n$ and $\partial\mathcal{G}^0 := \mathcal{V} \cap \mathbb{S}^{n-1}$. The set $\mathcal{G}$ is a ball with center p_0; let its radius be r. Then $\mathcal{G}$ and $\mathcal{G}^0$ are homeomorphic with the homeomorphism $h : \mathcal{G}^0 \to \mathcal{G}$ given by

$$h(z) = rz + p_0\,, \qquad z \in \mathcal{G}^0\,.$$

In other words, $\mathcal{G}^0$ can be taken to be a ball as well. Then we can define a continuous map $\widetilde{Y}^t : \mathcal{G}^0 \to \partial\mathcal{G}^0$ by

$$\widetilde{Y}^t(z) := \frac{Y^t(h(z))}{\|Y^t(h(z))\|}\,, \qquad z \in \mathcal{G}^0\,.$$

Let $\widetilde{y}^t = \partial\widetilde{Y}^t : \partial\mathcal{G}^0 \to \partial\mathcal{G}^0$ be the boundary map. For $z \in \partial\mathcal{G}^0$, let $x := h(z) \in \partial\mathcal{G}$. From (4.3) and (A.4)

$$z \cdot \widetilde{y}^t(z) \;=\; \frac{(x - p_0) \cdot Y^t(x)}{r\|Y^t(x)\|} = \frac{x \cdot Y^t(x)}{r\|Y^t(x)\|} = \frac{x \cdot Y(x)}{r\|Y^t(x)\|} \leq 0\,, \quad z \in \partial\mathcal{G}^0\,.$$

By Lemma 9, $\deg(\widetilde{y}^t) = (-1)^\kappa$, and by Theorem 10, $\widetilde{y}^t$ is not extendible to a continuous map $\widetilde{Y}^t : \mathcal{G}^0 \to \partial\mathcal{G}^0$. This gives a contradiction.

We conclude there exists $\overline{x} \in \mathcal{G}$ such that $Y^t(\overline{x}) = 0$. Using (A.5), $Y(\overline{x}) = Y^r(\overline{x}) + Y^c(\overline{x})$. By definition, $Y(\overline{x}) \in \mathcal{B}$.

Using (A.2) and that $(x \cdot x') \leq 1$ for all $x, x' \in \mathcal{G}$ from the Cauchy-Schwartz inequality, we have that for all $x \in \partial\mathcal{G}$

$$\begin{aligned} x \cdot (\overline{x} - p_1) &= x \cdot (\overline{x} - x) + x \cdot (x - p_1) \\ &= (x \cdot \overline{x}) - 1 \\ &\leq 0\,. \end{aligned} \tag{A.6}$$

Now suppose

$$Y^r(\overline{x}) = \overline{c}(\overline{x} - p_1)\,, \qquad \overline{c} \geq 0\,.$$

Using (A.6) and (A.3), we get

$$x \cdot \big(Y^r(\overline{x}) + Y^c(\overline{x})\big) = x \cdot Y(\overline{x}) \leq 0\,, \qquad x \in \partial\mathcal{G}\,.$$

From (4.4) we conclude $Y(\overline{x}) \in \mathcal{B} \cap \mathrm{cone}(\mathcal{G})$. Alternatively, if $\overline{c} < 0$, then the argument is repeated to get $-Y(\overline{x}) \in \mathcal{B} \cap \mathrm{cone}(\mathcal{G})$. By assumption $\mathcal{B} \cap \mathrm{cone}(\mathcal{G}) = \mathbf{0}$, so $Y(\overline{x}) = 0$. This is a contradiction since $Y : \mathcal{G} \to \mathcal{B}_1$ and $0 \notin \mathcal{B}_1$. $\square$