

The Maximax Minimax Quotient Theorem

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Abstract We present an optimization problem emerging from optimal control theory and situated at the intersection of fractional programming and linear max-min programming on polytopes. A naïve solution would require solving four nested, possibly nonlinear, optimization problems. Instead, relying on numerous geometric arguments we determine an analytical solution to this problem. In the course of proving our main theorem we also establish another optimization result stating that the minimum of a specific minimax optimization is located at a vertex of the constraint set.

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1 Introduction

The field of fractional programming studies the optimization of a ratio of functions and made its debut in the 1960s with Charnes and Cooper [4]. It has since then expanded to more complex and more general problems [9]. However, outside of linear fractional programming, very few analytical results

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are available; the focus has now largely shifted to developing search algorithms [1],[8]. In this paper we are interested in a specific fractional optimization problem introduced in [3] and composed of four nested optimization problems. For this reason, a search algorithm would have a high computational cost and would be especially wasteful since an analytical solution exists.

Our ratio of interest features a max-min optimization [5], [6] belonging to the setting of semi-infinite programming [7]. Because of the infinite number of constraints, it is not possible to immediately apply the classical results of linear max-min theory [10] stating that the maximum is attained on the boundary of the constraint set. Nonetheless, thanks to the specific geometry of our problem we are able to prove a very similar result, first mentioned as Theorem 3.1 in the authors' work [3]. However, its proof is omitted from [3].

Armed with this preliminary result on max-min programming, we formulate and establish the Maximax Minimax Quotient Theorem. This result concerns the maximization of a ratio of a maximum and a minimax over two polytopes. In the special case where these polytopes are symmetric, this result reduces to Theorem 3.2 of [3], whose proof was again omitted for length concerns.

The remainder of this paper is organized as follows. Section 2 establishes the existence of the Maximax Minimax Quotient and proves a preliminary optimization result. Section 3 states our central theorem and provides its proof. Section 4 gathers all the lemmas involved in the proof of the Maximax Minimax Quotient Theorem. Section 5 justifies the continuity of two maxima functions used during the proof of our main result. Finally, Section 6 illustrates the proof of our theorem on a simple example.

Notation: We use ∂X to denote the boundary of a set X and its interior is denoted $X^\circ := X \setminus \partial X$. In \mathbb{R}^n we denote the unit sphere with $\mathbb{S} := \{x \in \mathbb{R}^n : \|x\| = 1\}$ and the ball of radius ε centered on x with $B_\varepsilon(x) := \{y \in \mathbb{R}^n : \|y - x\| \leq \varepsilon\}$. The scalar product of vectors is denoted by $\langle \cdot, \cdot \rangle$. For $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ both nonzero we denote as $\widehat{x, y}$ the signed angle from x to y in the 2D plane containing both of them. We take the convention that the angles are positive when going in the clockwise orientation.

2 Preliminaries

Definition 2.1 A *polytope* in \mathbb{R}^n is a compact intersection of finitely many half-spaces.

Following Definition 2.1, this work only considers convex polytopes. If X and Y are two nonempty polytopes in \mathbb{R}^n with $-X \subset Y^\circ$, and $d \in \mathbb{S}$, we define the *Maximax Minimax Quotient* as

$$r_{X,Y}(d) := \frac{\max_{x \in X, y \in Y} \{\|x + y\| : x + y \in \mathbb{R}^+ d\}}{\min_{x \in X} \left\{ \max_{y \in Y} \{\|x + y\| : x + y \in \mathbb{R}^+ d\} \right\}}. \quad (1)$$

The objective of the Maximax Minimax Quotient Theorem is to determine the direction d that maximizes $r_{X,Y}(d)$. Note that in the numerator of (1), x and y are chosen together to satisfy the constraint $x + y \in \mathbb{R}^+d$, while in the denominator this constraint only applies to y . Before starting the actual proof of this theorem, we first need to justify the existence of the minimum and the maxima appearing in (1).

Proposition 2.1 *Let X, Y be two nonempty polytopes in \mathbb{R}^n with $-X \subset Y^\circ$, $\dim Y = n$ and $d \in \mathbb{S}$. Then,*

- (i) $\max_{x \in X, y \in Y} \{\|x + y\| : x + y \in \mathbb{R}^+d\}$ exists,
- (ii) $\lambda^*(x, d) := \max_{y \in Y} \{\|x + y\| : x + y \in \mathbb{R}^+d\}$ exists for all $x \in X$,
- (iii) $\min_{x \in X} \{\lambda^*(x, d)\}$ exists,
- (iv) and $\min_{x \in X} \{\lambda^*(x, d)\} > 0$.

Proof (i) Let $S := \{(x, y) \in X \times Y : x + y \in \mathbb{R}^+d\}$. Set S is a closed subset of the compact set $X \times Y$, so S is compact. Since X is nonempty, we take $x \in X$. Using $-X \subset Y$ we have $-x \in Y$ and $x + (-x) = 0 \in \mathbb{R}^+d$. Then, $(x, -x) \in S$, so S is nonempty. Function $f : S \rightarrow \mathbb{R}$ defined as $f(x, y) := \|x + y\|$ is continuous, so it reaches a maximum over S .

(ii) For $x \in X$ define $S(x) := \{y \in Y : x + y \in \mathbb{R}^+d\}$. Since $S(x)$ is a closed subset of the compact set Y , $S(x)$ is compact. Since $-X \subset Y$, we have $-x \in S(x)$ and so $S(x) \neq \emptyset$. Function $f_x : S(x) \rightarrow \mathbb{R}$ defined as $f_x(y) := \|x + y\|$ is continuous, so it reaches a maximum over $S(x)$, i.e., λ^* exists.

(iii) For $x \in X$ and $d \in \mathbb{S}$, the argument of $\lambda^*(x, d)$ is uniquely defined as $y^*(x, d) := \lambda^*(x, d)d - x$ since $\|d\| = 1$ and

$$y^*(x, d) = \arg \max_{y \in Y} \{\|x + y\| : x + y \in \mathbb{R}^+d\}. \quad (2)$$

Lemma 5.2 shows that λ^* is continuous in x and d , so y^* is also continuous in x and d . Then, function $f : X \rightarrow \mathbb{R}$ defined as $f(x) := \|x + y^*(x, d)\|$ is continuous, so it reaches a minimum over the compact and nonempty set X .

(iv) Note that $y^*(x, d) \in \partial Y$ for all $x \in X$. Indeed, assume for contradiction purposes that there exists $\varepsilon > 0$ such that $B_\varepsilon(y^*(x, d)) \subset Y$. We required $\dim Y = n$ to make this ball of full dimension, so that $z := y^*(x, d) + \varepsilon d \in Y$. Then, $x + z = (\lambda^*(x, d) + \varepsilon)d \in \mathbb{R}^+d$ and $\|x + z\| = \lambda^*(x, d) + \varepsilon > \lambda^*(x, d)$ contradicting the optimality of λ^* . Thus, $y^*(x, d) \in \partial Y$. Since $-X \subset Y^\circ$, we have $\|x + y^*(x, d)\| > 0$ for all $x \in X$. \square

Then, with the assumptions of Proposition 2.1 the Maximax Minimax Quotient is well-defined. The proof of our main theorem relies on another optimization result stating that the argument of the minimum in (1) lies at a vertex of X .

Definition 2.2 A *vertex* of a set $X \subset \mathbb{R}^n$ is a point $x \in X$ such that if there are $x_1 \in X$, $x_2 \in X$ and $\lambda \in [0, 1]$ with $x = \lambda x_1 + (1 - \lambda)x_2$, then $x = x_1 = x_2$.

With this definition, a vertex of a polytope corresponds to the usual understanding of a vertex of a polytope.

Theorem 2.1 Let $d \in \mathbb{S}$, X and Y two polytopes of \mathbb{R}^n with $-X \subset Y$ and $\dim Y = n$. Then, there exists a vertex v of X where $\min_{x \in X} \{\lambda^*(x, d)\}$ is reached.

Proof According to Proposition 2.1 the minimum of λ^* exists. Then, let $x^* \in X$ such that $\lambda^*(x^*, d) = \min_{x \in X} \{\lambda^*(x, d)\}$, i.e., $\|y^*(x^*) + x^*\| = \min_{x \in X} \|y^*(x) + x\|$. Since $-x^*$ must minimize the distance between itself and $y^*(x^*) \in \partial Y$, with $-X \subset Y$ obviously $x^* \in \partial X$. Assume now that x^* is not on a vertex of ∂X . Let S_x be the surface of lowest dimension in ∂X such that $x^* \in S_x$ and $\dim S_x \geq 1$.

Let v be a vertex of S_x and $x(\alpha) := x^* + \alpha(v - x^*)$ for $\alpha \in \mathbb{R}$. Notice that $x(0) = x^*$ and $x(1) = v$. Due to the choice of v , the convexity of S_x and x^* not being a vertex, there exists $\varepsilon > 0$ such that $x(\alpha) \in S_x$ for all $\alpha \in [-\varepsilon, 1]$. We also define the lengths $L(\alpha) := \|y^*(x(\alpha)) + x(\alpha)\|$ and $L^* := L(0)$.

Since $\|d\| = 1$ and $y^*(x(\alpha)) + x(\alpha) \in \mathbb{R}^+d$, we have $L(\alpha) = \langle y^*(x(\alpha)) + x(\alpha), d \rangle$. By definition of x^* , we know that $L^* \leq L(\alpha)$ for all $\alpha \in [-\varepsilon, 1]$. For contradiction purposes assume that there exists $\alpha_0 \in (0, 1]$ such that $L^* < L(\alpha_0)$. We introduce the convexity coefficient $\beta := \frac{\alpha_0}{\alpha_0 + \varepsilon} > 0$ and then

$$\begin{aligned} L^* &= \beta L^* + (1 - \beta)L^* < \beta L(-\varepsilon) + (1 - \beta)L(\alpha_0) \\ &< \beta \langle y^*(x(-\varepsilon)) + x(-\varepsilon), d \rangle + (1 - \beta) \langle y^*(x(\alpha_0)) + x(\alpha_0), d \rangle = \langle z + x^*, d \rangle, \end{aligned}$$

with $z := \beta y^*(x(-\varepsilon)) + (1 - \beta)y^*(x(\alpha_0))$. Indeed, note that $\beta x(-\varepsilon) + (1 - \beta)x(\alpha_0) = x^*$, and $z + x^* \in \mathbb{R}^+d$. Note that $L^* = \max_{y \in Y} \{\langle x^* + y, d \rangle : x^* + y \in \mathbb{R}^+d\}$, but $L^* < \langle x^* + z, d \rangle$. Given that $z \in Y$ by convexity of Y and $x^* + z \in \mathbb{R}^+d$, we have reached a contradiction. Thus, there is no $\alpha_0 \in (0, 1]$ such that $L^* < L(\alpha_0)$. Therefore, for all $\alpha \in [0, 1]$, $L(\alpha) = L^*$. By taking $\alpha = 1$, we have $x(\alpha) = v$, so the minimum L^* is also reached on the vertex v of X . \square

We have now all the preliminary results necessary to state our central theorem.

3 The Maximax Minimax Quotient Theorem

Theorem 3.1 (Maximax Minimax Quotient Theorem) If X and Y are two polytopes in \mathbb{R}^n with $-X \subset Y^\circ$, $\dim X = 1$, $\partial X = \{x_1, x_2\}$ with $x_2 \neq 0$ and $\dim Y = n$, then $\max_{d \in \mathbb{S}} r_{X,Y}(d) = \max \{r_{X,Y}(x_2), r_{X,Y}(-x_2)\}$.

Proof Since $\dim X = 1$, its extremities x_1 and x_2 are different, so at least one of them is nonzero. Then, imposing $x_2 \neq 0$ does not restrain the generality of our result. Following Proposition 2.1, $r_{X,Y}$ is well-defined. Reusing y^* defined in (2), we introduce $x_M^*(d) := \arg \min_{x \in X} \{\|x + y^*(x, d)\|\}$ and $x_N^*(d) := \arg \max_{x \in X} \{\|x + y^*(x, d)\| : x + y^*(x, d) \in \mathbb{R}^+ d\}$. According to Theorem 2.1, $x_M^*(d) \in \partial X$ for all $d \in \mathbb{S}$ and following Lemma 5.3, x_N^* is a continuous function of d . For some $d \in \mathbb{S}$ the arg min and arg max in the definitions of x_M^* and x_N^* might not be unique; if so we take the arguments ensuring that $x_M^*(d) \in \partial X$ and that x_N^* is continuous. We also define $y_N^*(d) := y^*(x_N^*(d), d)$ and $y_M^*(d) := y^*(x_M^*(d), d)$. Then,

$$r_{X,Y}(d) = \frac{\max_{y \in Y} \{\|y + x_N^*(d)\| : y + x_N^*(d) \in \mathbb{R}^+ d\}}{\max_{y \in Y} \{\|y + x_M^*(d)\| : y + x_M^*(d) \in \mathbb{R}^+ d\}} = \frac{\|x_N^*(d) + y_N^*(d)\|}{\|x_M^*(d) + y_M^*(d)\|}.$$

Since $\dim X = 1$, we can take \mathcal{P} to be a two-dimensional plane containing X . Then, we will study how $r_{X,Y}(d)$ varies when d takes values in $\mathbb{S} \cap \mathcal{P}$. We introduce the signed angles $\alpha := \widehat{d, \partial Y}$ and $\beta := \widehat{x_2, d}$. These angles are represented on Figure 1 and they take value in $[0, 2\pi)$. We parametrize all directions $d \in \mathbb{S} \cap \mathcal{P}$ by the angle β . Then, we will study how $r_{X,Y}(d)$ varies when $\beta \in [0, 2\pi)$.

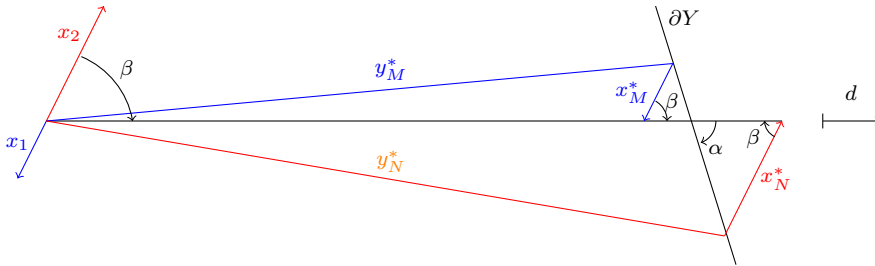


Fig. 1 Illustration of y_N^* , x_N^* , y_M^* and x_M^* for a direction d parametrized by β .

We first establish in Lemma 4.1 that $x_N^*(d)$ and $x_M^*(d)$ are constant, different and both belong in ∂X when $y_M^*(d)$, d and $y_N^*(d)$ all intersect the same face of ∂Y , as illustrated on Figure 1. In these situations, Lemma 4.2 shows that the ratio $r_{X,Y}$ is constant. Thus, $r_{X,Y}$ can only change when one of the three rays intersects a different face of ∂Y than the other two. We refer to these situations as vertex crossings. Lemma 4.3 introduces the vertices v_π and $v_{2\pi}$.

We study the crossing of vertices before v_π in Lemma 4.4. During these crossings Lemma 4.5 shows that $r_{X,Y}$ decreases as β increases. Lemma 4.6 states that $r_{X,Y}$ reaches a local minimum during the crossing of v_π . As β increases between v_π and π , Lemmas 4.7 and 4.8 prove that $r_{X,Y}$ increases

during the crossing of vertices. Finally, Lemma 4.9 completes the revolution by showing that $r_{X,Y}$ decreases after $\beta = \pi$ until a local minimum at $v_{2\pi}$ and then increases again until $\beta = 2\pi$. Thus, the directions $d \in \mathcal{P} \cap \mathbb{S}$ maximizing $r_{X,Y}(d)$ are collinear with the set X . Note that Figure 1 implicitly assumes that $0 \in X$. Lemma 4.10 proves that even if $0 \notin X$ all above results still hold. Therefore, $\max_{d \in \mathbb{S}} r_{X,Y}(d) = \max_{\mathcal{P}} \left\{ \max_{d \in \mathcal{P} \cap \mathbb{S}} r_{X,Y}(d) \right\} = \max \{ r_{X,Y}(x_2), r_{X,Y}(-x_2) \}$. \square

In the special case where X and Y are symmetric polytopes, this result reduces to Theorem 3.2 of [3]. Indeed, $r_{X,Y}$ becomes an even function which leads to $r_{X,Y}(x_2) = r_{X,Y}(-x_2)$.

4 Supporting Lemmata

In this section we establish all the lemmas involved in the proof of the Maximax Minimax Quotient Theorem.

Lemma 4.1 *If d , $y_N^*(d)$ and $y_M^*(d)$ all intersect the same face of ∂Y , then $x_N^*(d)$ and $x_M^*(d)$ are constant, different and both belong to ∂X .*

Proof We introduce the angles $\beta_M := \widehat{x_2, y_M^*}$ and $\beta_N := \widehat{x_2, y_N^*}$. Let α_0 be the value of α when $\beta = 0$, i.e., when d is positively collinear with x_2 .

We say that y_N^* is *leading* and y_M^* is *trailing* when $\beta_M < \beta_N$, and conversely when $\beta_N < \beta_M$, we say that y_M^* is *leading* and y_N^* is *trailing*.

For each $d \in \mathbb{S} \cap \mathcal{P}$ we define $D(d) := \max_{y \in Y} \{ \|y\| : y \in \mathbb{R}^+ d \}$, whose existence is justified by the compactness of Y .

We say that y_N^* or y_M^* is *outside* when $\|y_N^* + x_N^*\| > D$ or $\|y_M^* + x_M^*\| > D$ respectively. Otherwise, y_N^* or y_M^* is *inside*. Directly related to the previous definition, we introduce

$$\delta_M(d) := D(d) - \|x_M^*(d) + y_M^*(d)\| \quad \text{and} \quad \delta_N(d) := \|x_N^*(d) + y_N^*(d)\| - D(d). \quad (3)$$

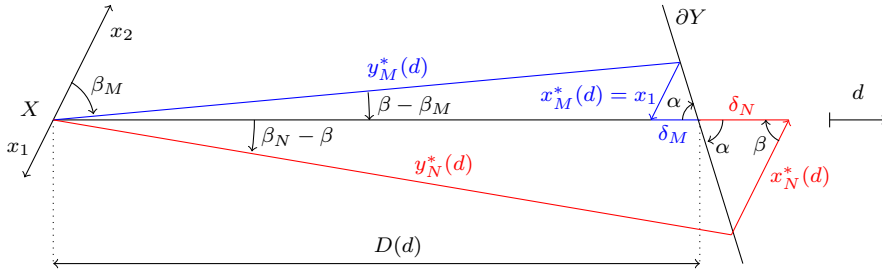


Fig. 2 Illustration of $y_N^*(d)$ leading and outside, while $y_M^*(d)$ is trailing and inside the same face of ∂Y .

We know from Theorem 2.1 that $x_M^*(d) \in \partial X$ for all $d \in \mathbb{S}$. In the case illustrated on Figure 2, $x_M^*(d) = x_1$ because it maximizes δ_M .

If $\alpha + \beta \in \{\pi, 2\pi\}$, then X is parallel with a face of ∂Y making x_N^* and x_M^* not uniquely defined. Regardless, we can still take $x_N^*(d) \neq x_M^*(d)$, with $x_N^*(d) \in \partial X$ and $x_M^*(d) \in \partial X$. Otherwise, x_N^* and x_M^* are uniquely defined. Since $x_N^*(d) \in X$, $x_M^*(d) \in X$ for all $d \in \mathbb{S}$ and $\dim X = 1$, vectors $x_N^*(d)$ and $x_M^*(d)$ are always collinear. We then use Thales's theorem and obtain $\delta_N(d) = \delta_M(d) \frac{\|x_N^*(d)\|}{\|x_M^*(d)\|}$. Since $x_N^*(d)$ is chosen to maximize δ_N and is independent from δ_M , it must have the greatest norm, so $x_N^*(d) \in \partial X$. In the case where $\alpha + \beta \notin \{\pi, 2\pi\}$, $\|x + y\|$ depends on the value of x . Because $x_N^*(d)$ is chosen to maximize $\|x + y\|$ while $x_M^*(d)$ is minimizing it, we have $x_N^*(d) \neq x_M^*(d)$.

Since x_N^* is continuous according to Lemma 5.3 and $x_N^*(d) \in \{x_1, x_2\}$, then $x_N^*(d)$ is constant on the faces of ∂Y . Because $x_M^*(d) \in \partial X$ too, it must also be constant. \square

Lemma 4.2 *When d , $y_N^*(d)$ and $y_M^*(d)$ all intersect the same face of ∂Y , the ratio $r_{X,Y}(d)$ is constant.*

Proof Based on Figure 2 we apply the sine law in the triangle bounded by ∂Y , δ_M and x_M^*

$$\frac{\|x_M^*(d)\|}{\sin \alpha} = \frac{\delta_M(d)}{\sin(\pi - \alpha - \beta)} = \frac{\delta_M(d)}{\sin(\alpha + \beta)}, \quad \text{so} \quad \frac{\delta_M(d)}{D(d)} = \frac{\|x_M^*(d)\| \sin(\alpha + \beta)}{D(d) \sin \alpha}.$$

Similarly for the triangle bounded by ∂Y , δ_N and x_N^* , the law of sines yields

$$\frac{\|x_N^*(d)\|}{\sin \alpha} = \frac{\delta_N(d)}{\sin(\pi - \alpha - \beta)} = \frac{\delta_N(d)}{\sin(\alpha + \beta)}, \quad \text{so} \quad \frac{\delta_N(d)}{D(d)} = \frac{\|x_N^*(d)\| \sin(\alpha + \beta)}{D(d) \sin \alpha}.$$

Even if the two equations above were derived for the specific situation of Figure 2, they hold as long as y_N^* , D and y_M^* intersect the same face of ∂Y . Based on (3) we have

$$r_{X,Y}(d) = \frac{D(d) + \delta_N(d)}{D(d) - \delta_M(d)} = \frac{1 + \frac{\delta_N}{D}}{1 - \frac{\delta_M}{D}}. \quad (4)$$

We will now prove that the ratios δ_N/D and δ_M/D do not change on a face of ∂Y . Let $d_1 \in \mathcal{P} \cap \mathbb{S}$ and $d_2 \in \mathcal{P} \cap \mathbb{S}$ such that $D(d_1)$, $D(d_2)$, $y_M^*(d_1)$, $y_M^*(d_2)$, $y_N^*(d_1)$ and $y_N^*(d_2)$ all intersect the same face of ∂Y , as illustrated on Figure 3.

The sum of the angles of the triangle in Figure 3 is

$$(\beta_2 - \beta_1) + \alpha_2 + (\pi - \alpha_1) = \pi \quad \text{so} \quad \beta_2 + \alpha_2 = \beta_1 + \alpha_1. \quad (5)$$

Therefore, $\alpha + \beta$ is constant on faces of ∂Y . We use the sine law in the triangle in Figure 3 and obtain

$$\frac{D(d_1)}{\sin \alpha_2} = \frac{D(d_2)}{\sin(\pi - \alpha_1)} = \frac{D(d_2)}{\sin \alpha_1}, \quad \text{so,} \quad D(d_1) \sin \alpha_1 = D(d_2) \sin \alpha_2.$$

According to Lemma 4.1 we also know that $x_N^*(d_1) = x_N^*(d_2)$, thus

$$\frac{\delta_N(d_1)}{D(d_1)} = \frac{\|x_N^*(d_1)\| \sin(\alpha_1 + \beta_1)}{D(d_1) \sin \alpha_1} = \frac{\|x_N^*(d_2)\| \sin(\alpha_2 + \beta_2)}{D(d_2) \sin \alpha_2} = \frac{\delta_N(d_2)}{D(d_2)}.$$

The same holds for δ_M/D . Hence, (4) yields $r_{X,Y}(d_1) = r_{X,Y}(d_2)$. \square

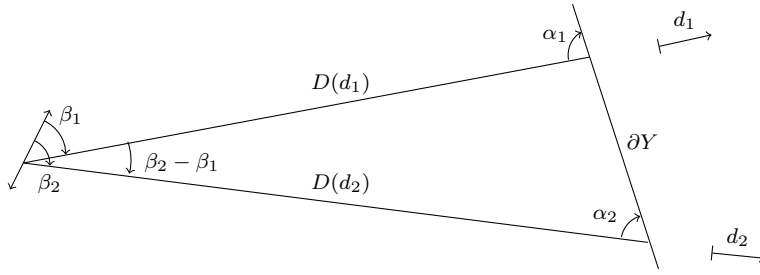


Fig. 3 Ratio $r_{X,Y}(d)$ is constant on a face of ∂Y .

Lemma 4.3 *There are two vertices of $Y \cap \mathcal{P}$, namely v_π and $v_{2\pi}$ whose crossing by d makes the angle $\alpha + \beta$ become greater than π and 2π respectively.*

Proof We have taken the convention that the angles are positively oriented in the clockwise orientation. According to (5), the angle $\alpha + \beta$ is constant on a face of ∂Y . When d crosses a vertex of external angle ε as represented on Figure 5, the value of α has a discontinuity of $+\varepsilon$. Let q be the number of vertices of ∂Y and ε_i the external angle of the i^{th} vertex v_i . Since $Y \cap \mathcal{P}$ is a polygon, $\sum_{i=1}^q \varepsilon_i = 2\pi$. We can then represent the evolution of $\alpha + \beta$ as a function of β with Figure 4. Instead of labeling the horizontal axis with the values taken by β as the corresponding vector $d(\beta)$ crosses the vertex v_i , we directly use v_i with a slight abuse of notation.

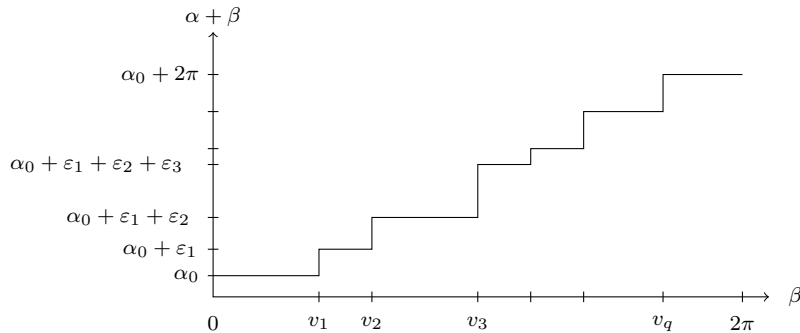


Fig. 4 Evolution of $\alpha + \beta$ with β increasing clockwise in $[0, 2\pi)$.

Recall that α_0 is the value of α when $\beta = 0$. After a whole revolution $\alpha + \beta = \alpha_0 + 2\pi$. So there are two vertices v_π and $v_{2\pi}$ where $\alpha + \beta$ first crosses π and then 2π . In the eventuality that $\alpha + \beta = \pi$ or 2π on a face of ∂Y , we define v_π or $v_{2\pi}$ as the vertex preceding the face. \square

Lemma 4.4 *During the crossing of vertices before v_π as β increases, $x_N^*(d) = x_2$ and $x_M^*(d) = x_1$. They are constant, different and both belong in ∂X .*

Proof We study the crossing of a vertex v of angle ε between the faces F_1 and F_2 of ∂Y . For each vertex v we introduce x_v the vector collinear with X , going from v to the ray directed by d , as illustrated on Figure 5 and we say that the crossing of v is ongoing as long as $\|x_v\| < \max\{\|x_1\|, \|x_2\|\}$. We also define $\delta_v := \|v + x_v\| - D$.

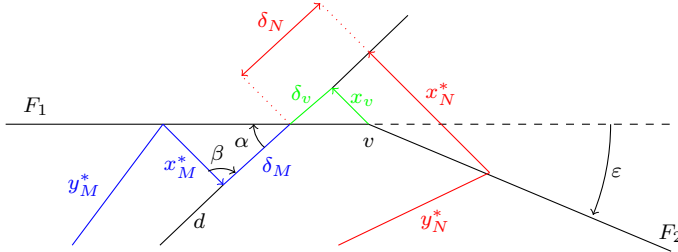


Fig. 5 Illustration of x_v during the crossing of a vertex v , with y_N^* leading.

Before starting the crossing of v_π we have $\alpha + \beta \in (\alpha_0, \pi)$. This situation is depicted on Figure 2, where y_N^* is leading and outside, so y_N^* reaches the vertex before y_M^* and d . The length of $x_N^*(d)$ can vary to maximize δ_N , so y_N^* could still intersect F_1 , even if the crossing is ongoing. We have seen in Lemma 4.1 that if y_N^* is still on F_1 , then it must be the furthest possible to maximize δ_N , in that case $y_N^* = v$. Otherwise, y_N^* intersects F_2 . We want to establish a criterion to distinguish these two possible scenarios.

We first consider the scenario where $y_N^* = v$ and $x_N^*(d) = x_v$. We take $y \in F_2 \setminus \{v\}$ such that $x_2 + y \in \mathbb{R}^+ d$ as represented on Figure 6 and we define $\delta := \|x_2 + y\| - D$.

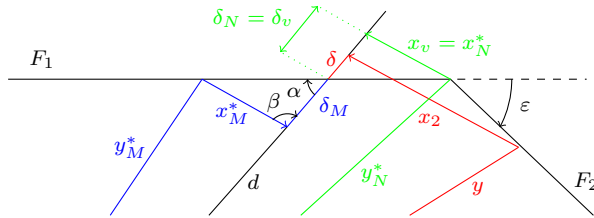


Fig. 6 Illustration of the crossing scenario where $y_N^* = v$.

Since δ_N must be maximized by the choice of y_N^* and $y \neq y_N^*$, we have $\delta < \delta_N = \delta_v$. But $\|x_2\| > \|x_v\|$, so the line segment corresponding to x_2 crosses the interior of Y . Focusing on this part of Figure 6 we obtain Figure 7.

Two of the angles of the triangle delimited by F_1 , F_2 and x_2 are $\pi - \alpha - \beta$ and $\pi - \varepsilon$. Therefore, their sum is in $(0, \pi)$ and thus $\alpha + \beta + \varepsilon > \pi$. Since we assumed that $\alpha + \beta \in (\alpha_0, \pi)$, the vertex v must in fact be v_π for this scenario to happen.

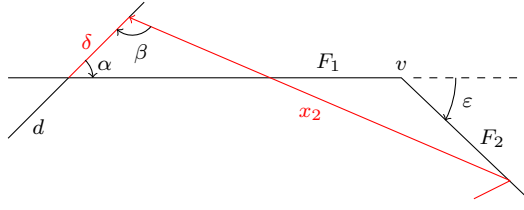


Fig. 7 Illustration of the line segment corresponding to x_2 crossing the interior of Y in Figure 6.

Thus, the crossing of a vertex preceding v_π follows the second scenario as depicted on Figure 5 with $y_N^* \in F_2$. We study Figure 8 which is a more detailed view of Figure 5, with δ_0 depending solely on d and ε .

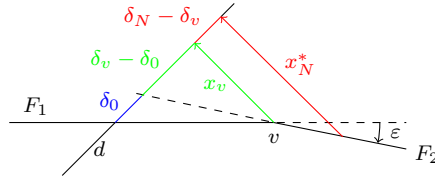


Fig. 8 Illustration of x_v and x_N^* in Figure 5.

Since x_v and $x_N^*(d)$ are collinear, we can apply Thales's theorem in Figure 8 and obtain that $\delta_N - \delta_0 = (\delta_v - \delta_0) \frac{\|x_N^*(d)\|}{\|x_v(d)\|}$. Then, δ_N is maximized when $\|x_N^*(d)\|$ is maximal, so $x_N^*(d) = x_2$ during the crossing. We know from Theorem 2.1 that $x_M^*(d) \in \partial X$ for all $d \in \mathbb{S}$. Then, as in Lemma 4.1, x_N^* and x_M^* are constant and different since x_N^* is continuous in d , so $x_M^*(d) = x_1$. \square

Lemma 4.5 *During the crossing of vertices before v_π as β increases, $r_{X,Y}(d)$ decreases.*

Proof The leading vector y_N^* is outside and crosses a vertex v between the faces F_1 and F_2 of ∂Y while β increases. We separate the vertex crossing into two parts: when only $y_N^* \in F_2$, and when both $d \in F_2$ and $y_N^* \in F_2$. Let $\varepsilon > 0$ be the external angle of the vertex as shown on Figure 9.

According to Lemma 4.2, $r_{X,Y}$ is constant on faces of ∂Y and we call r_{F_1} its value on the face F_1 . If F_1 was prolonged past v with a straight line (dashed line on Figure 9), then we would have $y_N^*(d) \in F_1$ and $r_{X,Y}(d) = r_{F_1}$. But, $y_N^*(d) \in F_2$ as proven in Lemma 4.4 because the crossing occurs before v_π . We call l the resulting difference in δ_N as illustrated on Figure 9. Notice that the two green segments of length l in Figure 9 are parallel. We parametrize the position of y_N^* on F_2 with the length m as defined on Figure 9. When $y_N^* = v$, $m = 0$, and m increases with β . Using the sine law we obtain

$$\frac{m}{\sin \beta} = \frac{l}{\sin(\pi - \alpha - \beta)} = \frac{l}{\sin(\alpha + \beta)}. \quad (6)$$

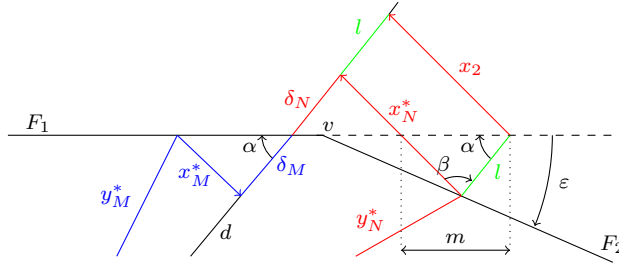


Fig. 9 Part I of the crossing of vertex v by y_N^* leading and outside as β increases.

Then,

$$r_{X,Y}(d) = \frac{D + \delta_N}{D - \delta_M} = \frac{D + \delta_N + l}{D - \delta_M} - \frac{l}{D - \delta_M} = r_{F_1} - \frac{m \sin(\alpha + \beta)}{(D - \delta_M) \sin(\beta)}. \quad (7)$$

By definition the length m is positive. Since $-x_M^* \in Y^\circ$ but $y_M^* \in \partial Y$, we have $D - \delta_M = \|y_M^* + x_M^*\| > 0$. Before v_π we have $\alpha + \beta \in (\alpha_0, \pi)$. In that case $\sin(\alpha + \beta) > 0$ and $\sin(\beta) > 0$. Therefore, the term subtracted from r_{F_1} is positive, i.e., $r_{X,Y}(d) < r_{F_1}$.

We can now tackle the second part of the crossing, when y_N^* and d both have crossed the vertex as illustrated on Figure 10.

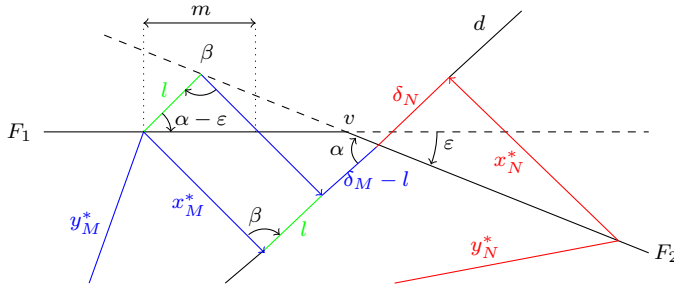


Fig. 10 Part II of the crossing of vertex v by y_N^* leading and outside as β increases.

If F_2 was prolonged with a straight line before v and $y_M^* \in F_2$, then we would have $r_{X,Y}(d) = r_{F_2}$, value of $r_{X,Y}$ on F_2 . But that is not the case, $y_M^*(d) \in F_1$ and the resulting difference in δ_M is called l . Using the sine law in Figure 10, we can relate l to m

$$\frac{m}{\sin \beta} = \frac{l}{\sin(\pi - \beta - \alpha + \epsilon)} = \frac{l}{\sin(\alpha + \beta - \epsilon)}. \quad (8)$$

We have $\alpha + \beta \in (\alpha_0, \pi)$, so $\sin(\beta) > 0$. If α was still measured between d and F_1 , then its value would be $\alpha_{F_1} = \alpha - \epsilon$. Since we are before the crossing of

v_π , $\alpha_{F_1} + \beta \in (\alpha_0, \pi)$, i.e., $\alpha + \beta - \varepsilon \in (\alpha_0, \pi)$. This yields $\sin(\alpha + \beta - \varepsilon) > 0$, which makes $l > 0$, because the length m is positive by definition. Then,

$$r_{F_2} = \frac{D + \delta_N}{D - (\delta_M - l)} = \frac{D + \delta_N}{D - \delta_M + l} < \frac{D + \delta_N}{D - \delta_M} = r_{X,Y}(d). \quad (9)$$

Thus, the ratio $r_{X,Y}$ decreases during the crossing of a vertex before v_π . \square

Lemma 4.6 *During the crossing of v_π , the ratio $r_{X,Y}(d)$ reaches a local minimum.*

Proof Recall that before the crossing, $x_N^*(d) = x_2$ and $x_M^*(d) = x_1$. During the crossing of v_π , i.e., when $\|x_{v_\pi}\| < \max\{\|x_1\|, \|x_2\|\}$, we have $\alpha + \beta \leq \pi$ but $\alpha + \beta + \varepsilon > \pi$. The situation is illustrated on Figure 11. We showed in Lemma 4.4 that $y_N^* = v_\pi$ and $x_N^*(d) = x_{v_\pi}$.

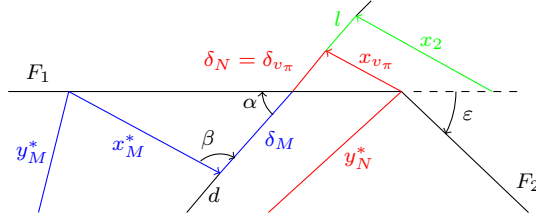


Fig. 11 Crossing of v_π , with $y_N^* = v_\pi$.

If F_1 was prolonged with a straight line (dashed line of Figure 11), we would have $y_N^* \neq v_\pi$, $x_N^*(d) = x_2$ and the ratio would be $r_{F_1} = \frac{D + \delta_{v_\pi} + l}{D - \delta_M}$, which is the value of $r_{X,Y}$ on F_1 . Since d has not yet crossed v_π , $\alpha + \beta < \pi$ and thus (6) and (7) still hold, leading to $r_{X,Y}(d) < r_{F_1}$.

Once d has crossed v_π , we still have $y_N^* = v_\pi$ to maximize δ_N . Then, the equality $x_N^*(d) = x_{v_\pi}$ holds during the whole crossing, i.e., as x_{v_π} goes from x_2 to x_1 . The second part of the crossing is illustrated on Figure 12.

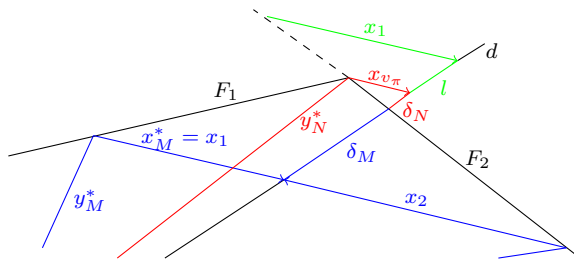


Fig. 12 Illustration of the endpoint of y_M^* switching from F_1 to F_2 during the crossing of v_π .

Assume that during the entire crossing of v_π , $x_M^*(d) = x_1$. Then, at the end of the crossing we will have $y_M^* = v_\pi$ and $x_M^*(d) = x_{v_\pi} = x_N^*(d)$, which contradicts the definitions of $x_M^*(d)$ and $x_N^*(d)$, they must be different. Thus, $x_M^*(d)$ does not remain equal to x_1 during the entire crossing. Since $x_M^* \in \{x_1, x_2\}$, at some point x_M^* switches to x_2 as y_M^* switches from F_1 to F_2 . This switching point is illustrated on Figure 12, and y_M^* becomes the leading vector.

After this switch, $y_M^* \in F_2$ and $x_M^*(d) = x_2$. If F_2 was prolonged with the dashed line on Figure 12, we would have $x_N^* = x_1$ instead of x_{v_π} with a gain of l for δ_N making the ratio equal to $r_{F_2} = \frac{D+\delta_N+l}{D-\delta_M}$, value of $r_{X,Y}$ on F_2 . But $x_N^* = x_{v_\pi}$ and $l > 0$, thus $r_{F_2} > \frac{D+\delta_N}{D-\delta_M} = r_{X,Y}(d)$. Therefore, $r_{X,Y}$ reaches a local minimum during the crossing of v_π . \square

Lemma 4.7 *During the crossing of vertices after v_π as β increases until π , $x_N^*(d) = x_1$ and $x_M^*(d) = x_2$. They are constant different and both in ∂X .*

Proof After the crossing of v_π , $\alpha + \beta \in (\pi, \alpha_0 + \pi)$ and y_M^* is leading and inside as established in Lemma 4.6. Thus, y_M^* is the first to reach vertex v . Since $x_M^* \in \{x_1, x_2\}$ we cannot have $x_M^* = x_v$ during the entire crossing because x_v is a continuous function of β . Thus y_M^* passes v and belongs to F_2 . In Lemma 5.3 we showed that x_N^* is continuous in d . Thus, $x_N^*(d)$ cannot switch like $x_M^*(d)$ did around v_π to take the lead. Instead, $x_N^*(d)$ is trailing as illustrated on Figure 13.

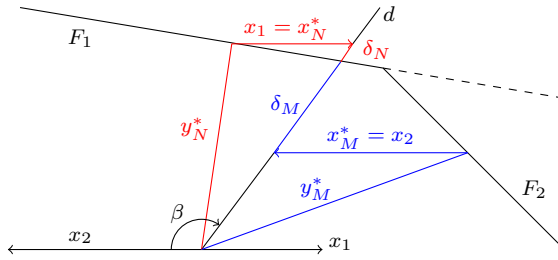


Fig. 13 Crossing of a vertex v after v_π .

Since $y_N^* \in F_1$ during the crossing, we can apply Thales's theorem on Figure 13 and obtain that for a fixed d , δ_N is proportional to $\|x_N^*(d)\|$. Thus, to maximize δ_N we have $x_N^*(d) \in \partial X$ and, since y_N^* is trailing, we have $x_N^*(d) = x_1$ during the entire crossing. By the definitions of $x_N^*(d)$ and $x_M^*(d)$, we have $x_N^*(d) \neq x_M^*(d)$. Since both $x_N^*(d)$ and $x_M^*(d)$ belong to $\partial X = \{x_1, x_2\}$, then $x_M^*(d) = x_2$ during the entire crossing. \square

Lemma 4.8 *During the crossing of vertices after v_π as β increases until π , $r_{X,Y}(d)$ increases.*

Proof The leading vector y_M^* is inside and crosses a vertex v between faces F_1 and F_2 as β increases. We define $\beta' := \pi - \beta$. Then, reversing the crossing

illustrated on Figure 13 is exactly the crossing illustrated on Figure 9 with β' increasing and x_1 and x_2 exchanged. According to Lemma 4.5, in that reversed crossing $r_{X,Y}$ is decreasing. Therefore, $r_{X,Y}$ increases during the crossing of vertices after v_π as β increases until π . \square

Lemma 4.9 *For $\beta > \pi$, $r_{X,Y}(d)$ decreases until $v_{2\pi}$ where it reaches a local minimum. After $v_{2\pi}$ as β increases until 2π , $r_{X,Y}(d)$ increases.*

Proof Let us change the angle convention, so that angles are now positively oriented in the counterclockwise orientation. The vertex that was previously labeled as $v_{2\pi}$ becomes the new v_π . Then, we only need to apply Lemmas 4.4, 4.5, 4.6, 4.7 and 4.8 to this new configuration to conclude the proof. \square

Lemma 4.10 *All above results hold even if $0 \notin X$.*

Proof In all the figures we made the implicit assumption that $0 \in X$, so that x_1 and x_2 were negatively collinear. Let x_1 be positively collinear with x_2 and $\|x_2\| > \|x_1\|$.

On Figure 2, we would now have $y_N^*(d)$ and $y_M^*(d)$ both outside. Then, the definition of δ_M should be adapted. Let $\delta_M(d) := \|x_M^*(d) + y_M^*(d)\| - D(d)$ and then $r_{X,Y}(d) = \frac{D+\delta_N}{D+\delta_M}$. Except for this modification, we would still have $x_N^*(d) = x_2$ and $x_M^*(d) = x_1$. Thales theorem can be used similarly to show that $x_N^*(d) \in \partial X$. Therefore, Lemma 4.1 holds.

In the proof of Lemma 4.2 we still have δ_N/D and δ_M/D invariant with respect to d on a given face of ∂Y , so $r_{X,Y}$ is still constant on faces. Lemma 4.3 is not affected at all. The first part of the crossing of a vertex before v_π as β increases is illustrated by Figure 14.

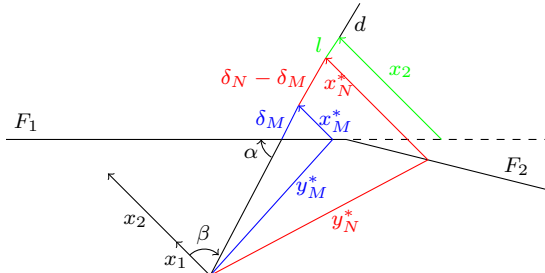


Fig. 14 Part I of the crossing of a vertex before v_π with $0 \notin X$.

For δ_M to be minimized and δ_N to be maximized, the Thales theorem clearly proves that $x_M^* \in \partial X$ and $x_N^* \in \partial X$ during the crossing. We still have $x_N^*(d) = x_2$ and $x_M^*(d) = x_1$, so Lemma 4.4 holds.

Following the reasoning in Lemma 4.5, we have $l > 0$, which leads to

$$r_{F_1} = \frac{D + \delta_N + l}{D + \delta_M} > \frac{D + \delta_N}{D + \delta_M} = r_{X,Y}(d).$$

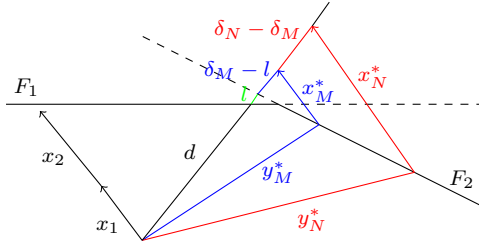


Fig. 15 Part II of the crossing of a vertex before v_π with $0 \notin X$.

During the second part, both $y_N^* \in F_2$ and $y_M^* \in F_2$ but $d \in F_1$. This situation is illustrated on Figure 15.

We compare the current value of $r_{X,Y}(d)$ with r_{F_2} , its value on F_2 :

$$r_{F_2} = \frac{D + (\delta_N - l)}{D + (\delta_M - l)} \quad \text{and} \quad r_{X,Y}(d) = \frac{D + \delta_N}{D + \delta_M}.$$

Since $l > 0$ and $\delta_N > \delta_M$, a simple calculation shows that $r_{X,Y}(d) < r_{F_2}$. Therefore, $r_{X,Y}$ is decreasing during the crossing of a vertex before v_π as β increases, Lemma 4.5 holds.

During the crossing of v_π , $y_N^* = v_\pi$ and $x_N^* = x_{v_\pi}$ with its norm decreasing continuously until $x_N^* = x_1$, while x_M^* will switch to x_2 in order to minimize δ_M . This is the same process as described in Lemma 4.6, so $r_{X,Y}$ also reaches a local minimum.

Because all the results studied so far still hold, then Lemmas 4.7, 4.8 and 4.9 hold too because they rely on those earlier results. \square

We have now established all the lemmas directly involved in the proof of the Maximax Minimax Quotient Theorem, but we still have a few claims of continuity to prove.

5 Continuity of Extrema

In Proposition 2.1 (iii) we needed the continuity of λ^* to prove it has a minimum and in Lemma 4.1 we used the continuity of x_N^* and y_N^* . In this section we will thus prove the continuity of these two maxima functions relying on the Berge Maximum Theorem [2].

Lemma 5.1 *Let X and Y be two nonempty polytopes in \mathbb{R}^n with $-X \subset Y$. Then, the set-valued function $\varphi : X \times \mathbb{S} \rightrightarrows Y$ defined as $\varphi(x, d) := Y \cap \{\lambda d - x : \lambda \geq 0\}$ satisfies Definition 17.2 of [2].*

Proof We define $\Omega := X \times \mathbb{S}$, so that $\varphi : \Omega \rightrightarrows Y$. On the space Ω we introduce the norm $\|\cdot\|_\Omega$ as $\|(x, d)\|_\Omega = \|x\| + \|d\|$. Since $\|\cdot\|$ is the Euclidean norm, $\|\cdot\|_\Omega$ is a norm on Ω . By Definition 17.2 of [2], we need to prove that φ is both upper and lower hemicontinuous at all points of Ω .

First, using Lemma 17.5 of [2] we will prove that φ is lower hemicontinuous by showing that for an open subset A of Y , $\varphi^l(A)$ is open. The lower inverse image of A is defined in [2] as

$$\begin{aligned}\varphi^l(A) &:= \{\omega \in \Omega : \varphi(\omega) \cap A \neq \emptyset\} \\ &= \{(x, d) \in X \times \mathbb{S} : Y \cap \{\lambda d - x : \lambda \geq 0\} \cap A \neq \emptyset\} \\ &= \{(x, d) \in X \times \mathbb{S} : \{\lambda d - x : \lambda \geq 0\} \cap A \neq \emptyset\},\end{aligned}$$

because $A \subset Y$. Let $\omega = (x, d) \in \varphi^l(A)$. Then, there exists $\lambda \geq 0$ such that $\lambda d - x \in A$. Since A is open, there exists $\varepsilon > 0$ such that the ball $B_\varepsilon(\lambda d - x) \subset A$. Now let $\omega_1 = (x_1, d_1) \in \Omega$ and denote $\varepsilon_x := \|x_1 - x\|$ and $\varepsilon_d := \|d_1 - d\|$. Then,

$$\|\lambda d_1 - x_1 - (\lambda d - x)\| = \|\lambda(d_1 - d) - (x_1 - x)\| \leq \lambda \varepsilon_d + \varepsilon_x.$$

Since $\lambda \geq 0$ is fixed, we can choose ε_d and ε_x positive and small enough so that $\lambda \varepsilon_d + \varepsilon_x \leq \varepsilon$. Then, we have showed that for all $\omega_1 = (x_1, d_1) \in \Omega$ such that $\|\omega - \omega_1\|_\Omega \leq \min(\varepsilon_d, \varepsilon_x)$, i.e., such that $\|x_1 - x\| \leq \varepsilon_x$ and $\|d_1 - d\| \leq \varepsilon_d$, we have $\lambda d_1 - x_1 \in B_\varepsilon(\lambda d - x) \subset A$, i.e., $\omega_1 \in \varphi^l(A)$. Therefore, $\varphi^l(A)$ is open, and so φ is lower hemicontinuous.

To prove the upper hemicontinuity of φ , we will use Lemma 17.4 of [2] and prove that for a closed subset A of Y , the lower inverse image of A is closed. Let $\{\omega_k\}$ be a sequence in $\varphi^l(A)$ converging to $\omega = (x, d) \in \Omega$. We want to prove that the limit $\omega \in \varphi^l(A)$.

For $k \geq 0$, we have $\omega_k = (x_k, d_k)$ and define $A_k := \{\lambda_k \geq 0 : \lambda_k d_k - x_k \in A\} \neq \emptyset$. Since A is a closed subset of the compact set Y , then A is compact. Thus A_k has a minimum and a maximum; we denote them by λ_k^{\min} and λ_k^{\max} respectively.

Since sequences $\{d_k\}$ and $\{x_k\}$ converge, they are bounded. The set A is also bounded, thus sequence $\{\lambda_k^{\max}\}$ is bounded. Let $\lambda^{\max} := \sup_{k \geq 0} \lambda_k^{\max} > 0$.

For $k \geq 0$, we define segments $S_k := \{\lambda d_k - x_k : \lambda \in [0, \lambda^{\max}]\}$, and $S := \{\lambda d - x : \lambda \in [0, \lambda^{\max}]\}$. These segments are all compact sets. We also introduce the sequences $a_k := \lambda_k^{\min} d_k - x_k \in A \cap S_k$ and $b_k := \lambda_k^{\min} d - x \in S$.

Take $\varepsilon > 0$. Since sequences $\{d_k\}$ and $\{x_k\}$ converge toward d and x respectively, there exists $N \geq 0$ such that for $k \geq N$, we have $\|d_k - d\| \leq \frac{\varepsilon}{2\lambda^{\max}}$ and $\|x_k - x\| \leq \frac{\varepsilon}{2}$. Then, for any $\lambda_k \in [0, \lambda^{\max}]$ as

$$\|\lambda_k d_k - x_k - (\lambda_k d - x)\| = \|\lambda_k(d_k - d) - (x_k - x)\| \leq \lambda_k \frac{\varepsilon}{2\lambda^{\max}} + \frac{\varepsilon}{2} \leq \varepsilon.$$

Since $\lambda_k^{\min} \in [0, \lambda^{\max}]$, we have $\|a_k - b_k\| \xrightarrow[k \rightarrow \infty]{} 0$. We define the distance between the sets A and S

$$\text{dist}(A, S) := \min \{\|a - s_\lambda\| : a \in A, s_\lambda \in S\}.$$

The minimum exists because A and S are both compact and the norm is continuous. Since $a_k \in A$ and $b_k \in S$, we have $\text{dist}(A, S) \leq \|a_k - b_k\|$ for all

$k \geq 0$. Therefore, $\text{dist}(A, S) = 0$. So, $A \cap S \neq \emptyset$, leading to $\omega \in \varphi^l(A)$. Then, $\varphi^l(A)$ is closed and so φ is upper hemicontinuous. \square

Lemma 5.2 *Let X and Y be two nonempty polytopes in \mathbb{R}^n with $-X \subset Y$. Then, $\lambda^*(x, d) := \max_{y \in Y} \{\|x + y\| : x + y \in \mathbb{R}^+d\}$ is continuous in $x \in X$ and $d \in \mathbb{S}$.*

Proof According to Proposition 2.1 (ii), whose proof does not rely on the current lemma, λ^* is well-defined. We introduce the set-valued function $\varphi : X \times \mathbb{S} \rightrightarrows Y$ defined by $\varphi(x, d) := \{y \in Y : x + y \in \mathbb{R}^+d\} = Y \cap (\mathbb{R}^+d - \{x\})$, where $\mathbb{R}^+d - \{x\} = \{\lambda d - x : \lambda \geq 0\}$.

We define the graph of φ as $\text{Gr } \varphi := \{(x, d, y) \in X \times \mathbb{S} \times Y : y \in \varphi(x, d)\}$, and the continuous function $f : \text{Gr } \varphi \rightarrow \mathbb{R}^+$ as $f(x, d, y) = \|x + y\|$. Set $X \times \mathbb{S}$ is compact and nonempty. Since Y is compact and $\mathbb{R}^+d - \{x\}$ is closed, their intersection $\varphi(x, d)$ is compact. Because $-X \subset Y$, for all $x \in X$ we have $-x \in \varphi(x, d)$, so $\varphi(x, d) \neq \emptyset$. According to Lemma 5.1, φ satisfies Definition 17.2 of [2]. Then, we can apply the Berge Maximum Theorem [2] and conclude that λ^* is continuous in x and d . \square

Lemma 5.3 *Let X and Y be two nonempty polytopes in \mathbb{R}^n with $-X \subset Y$. Then, the functions $(x_N^*, y_N^*)(d) = \arg \max_{x \in X, y \in Y} \{\|x + y\| : x + y \in \mathbb{R}^+d\}$ are continuous in $d \in \mathbb{S}$.*

Proof Let $Z := X + Y = \{x + y : x \in X, y \in Y\}$. Then Z is the Minkowski sum of two polytopes, so it is also a polytope [11]. According to Proposition 2.1 (i), whose proof does not rely on the current lemma, $\max_{x \in X, y \in Y} \{\|x + y\| : x + y \in \mathbb{R}^+d\}$ exists and thus $\max_{z \in Z} \{\|z\| : z \in \mathbb{R}^+d\}$ is also well-defined.

Since $-X \subset Y$, for all $x \in X$, $-x \in Y$ and thus $0 \in Z$. Then, $\{0\}$ and Z are two polytopes in \mathbb{R}^n with $\pm 0 \in Z$. According to Lemma 5.2 the function $\lambda^*(0, d) := \max_{z \in Z} \{\|z + 0\| : z + 0 \in \mathbb{R}^+d\}$ is continuous in $d \in \mathbb{S}$.

Then, we define the continuous function $z(d) := \lambda^*(0, d)d \in Z$ for $d \in \mathbb{S}$. Note that $z(d) = \arg \max_{z \in Z} \{\|z\| : z \in \mathbb{R}^+d\} = (x_N^*, y_N^*)(d)$, so these functions are continuous. \square

6 Illustration

We will now illustrate the Maximax Minimax Quotient Theorem on a simple example. We consider polygon X delimited by the vertices $x_1 = (0, -0.5)$ and $x_2 = (0, 1)$ in \mathbb{R}^2 and polygon Y with vertices $(\pm 1, \pm 2)$ and $(\pm 3, 0)$ as represented on Figure 16.

Since $-X \subset Y^\circ$, $\dim X = 1$, $x_2 \neq 0$ and $\dim Y = 2$, the assumptions of the Maximax Minimax Quotient Theorem are satisfied. To illustrate the proof of the theorem, for all $d \in \mathbb{S}$ we define the angle $\beta := \widehat{x_2, d}$ positively oriented clockwise. We also enumerate the vertices in the clockwise direction and we

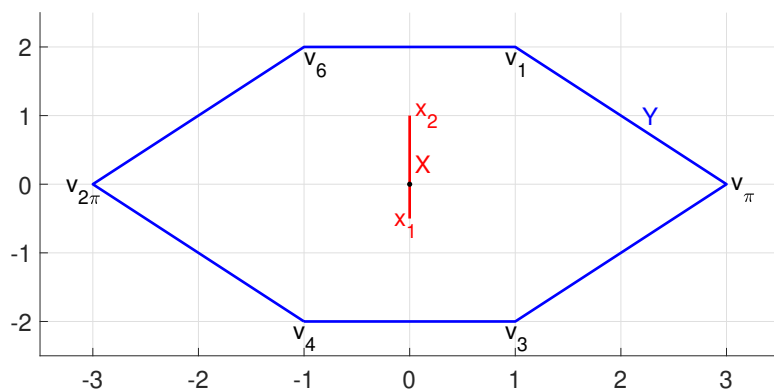


Fig. 16 Illustration of polygons X and Y .

note that $v_2 = v_\pi$ and $v_5 = v_{2\pi}$ as defined in Lemma 4.3. Then, we compute $r_{X,Y}$ for $\beta \in [0, 2\pi)$ as shown on Figure 17. The red spikes denote when the ray $d(\beta)$ hits a vertex of Y .

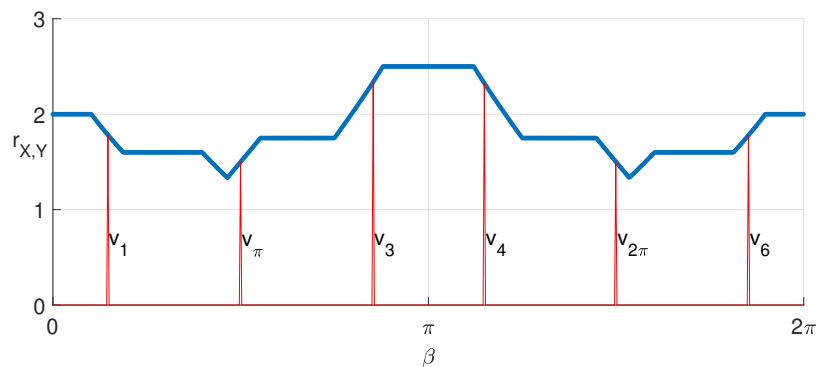


Fig. 17 Graph of $r_{X,Y}$ as a function of β .

As demonstrated by the Maximax Minimax Quotient Theorem, $r_{X,Y}$ has two local maxima achieved at $\beta = 0$ and $\beta = \pi$. These two values are different because polygon X is not symmetric. Note also that the Maximax Minimax Quotient Theorem does not state that the maximum is *only* reached when $\beta \in \{0, \pi\}$. Indeed as shown in Figure 17 and established in Lemma 4.2, $r_{X,Y}$ is constant on the faces of ∂Y . Thus, the two local maxima are achieved on the faces $[v_1, v_6]$ and $[v_3, v_4]$. As proven in Lemma 4.6 and in Lemma 4.9, $r_{X,Y}$ reaches a local minimum during the crossing of the vertices v_π and $v_{2\pi}$.

7 Conclusion

In this paper we considered an optimization problem arising from optimal control and pertaining to both fractional programming and max-min programming. We first justified the existence of the Maximax Minimax Quotient. Then, relying on numerous geometrical arguments and on the continuity of two maxima functions we were able to establish the Maximax Minimax Quotient Theorem. This result provides an analytical solution to the maximization of a ratio of a maximum and a minimax over two polytopes. We illustrated our theorem and its proof on a simple example in \mathbb{R}^2 . This work filled the theoretical gap left in [3], and because of our less restrictive assumptions we also open the way for a more general framework than that of [3]. A possible avenue for future work on this theorem is to study the case where $\dim X > 1$.

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