Losing Control of your Linear Network? Try Resilience Theory

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***Abstract*— Resilience of cyber-physical networks to unexpected failures is a critical need widely recognized across domains. For instance, power grids, telecommunication networks, transportation infrastructures and water treatment systems have all been subject to disruptive malfunctions and catastrophic cyber-attacks. Following such adverse events, we investigate scenarios where a node of a *linear* network suffers a loss of control authority over some of its actuators. These actuators are not following the controller's commands and are instead producing undesirable outputs. The repercussions of such a loss of control can propagate and destabilize the whole network despite the malfunction occurring at a single node. To assess system vulnerability, we establish resilience conditions for networks with a subsystem enduring a loss of control authority over some of its actuators. Furthermore, we quantify the destabilizing impact on the overall network when such a malfunction perturbs a nonresilient subsystem. We illustrate our resilience conditions on two academic examples, on an islanded microgrid, and on the linearized IEEE 39-bus system.**

***Index Terms*— Networked Control Systems, Resilience, Loss of Control, Cyber-Physical Systems.**

I. Introduction

RESILIENCE of cyber-physical networks to catastrophic events is a crucial challenge, widely recognized across government levels [1], [2] and research fields [3], [4]. Natural disasters, terrorist acts, and cyber-attacks all have the potential to paralyze the cyber-physical infrastructures upon which our society inconspicuously relies, such as power grids, telecommunication networks, sewage systems and transportation infrastructures [4]–[6]. Motivated by these issues, we investigate the resilience of *linear* networks to partial loss of control authority over their actuators. This class of malfunction, is characterized by some of the actuators producing uncontrolled and thus possibly undesirable outputs within their full capabilities [7]. This framework encompasses scenarios where actuators are taken over, for instance, by a cyber-attack [5], [6], and scenarios where actuators become unresponsive or damaged, for instance, by a software bug [8].

Building on fault-detection and isolation theory [9] coupled with cyber-attack detection [5] and state reconstruction methods [6], we assume that the controller has real-time readings of the outputs of the malfunctioning actuators. Our objective is then to assess the network's resilient stabilizability in the face of these possibly undesirable inputs [7].

All of our previous work on resilience theory [7], [10], etc., only investigated isolated systems enduring a partial loss of control authority over their actuators. When such a malfunctioning system is not isolated, but belongs instead to a network of interconnected systems, a loss of control can start a chain reaction capable of destabilizing the entire network. Such a problem has not been studied by previous resilience work and constitutes the main focus and novelty of this manuscript.

Albeit using a different setting, works [3], [11] also study the resilience of networks. Relying on observability and controllability, these works quantify the network's capabilities to detect a perturbed state and steer it back to its nominal value [3], [11]. Because the approach of such papers does not model the perturbation, it cannot handle a malfunctioning actuator producing undesirable inputs. Additionally, works [3], [6], [7], [11] require $\mathcal{L}_2$ inputs, whereas we are interested in component bounded inputs.

Traditionally, network resilience has been investigated through topological approaches [4], [12]–[15] using the network graph to reach a consensus between all nodes [12], [13], [16]. In this setting, after a loss of control authority over f nodes, at least $2f + 1$ disjoint paths are required for two nodes to exchange reliable information [12]. These works typically emphasize network architecture to the detriment of node dynamics, which are either unspecified [4], [15], [16], or restricted to a weighted average of neighbor states [12], [13], whereas we focus on networks of control systems with generic linear dynamics. Our control framework is also broader than the domain specific resilience studies focusing for instance on public transportation networks [4], Internet routing problems [15], or fluid transport networks [14].

In line with previous works studying actuator attacks [6], network cyber attacks [11], distributed consensus [12], [13], and power networks stability [17]–[20], we choose to focus on networks with linearized dynamics. The reader will realize that even with linear dynamics, the resilience of networks involves a copious amount of technical calculations.

The contributions of this work are threefold.

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1) We establish an *equivalence* condition to characterize resilient linear networks. This condition ensures that the network as a whole is resiliently stabilizable despite the loss of control authority over some actuators.
2) We *quantify the resilience* of fully-actuated networks having lost control over a nonresilient subsystem. More precisely, we calculate the maximal magnitude of undesirable inputs that the nonresilient subsystem can withstand without destabilizing the rest of the network by comparing the magnitude of perturbations due to subsystem couplings and their individual stability.
3) We extend the resilience quantification to *underactuated* networks losing control over a nonresilient subsystem. In this scenario, the malfunctioning subsystem prevents network stabilization but a feedback controller can maintain the network state within bounds.

The remainder of this paper is organized as follows. Section II introduces the network dynamics and states our problems of interest. Section III establishes stabilizability conditions for resilient linear networks. Section IV quantifies the resilient stabilizability of networks losing control authority over nonresilient subsystems. We illustrate our work on two academic examples, on an islanded microgrid, and on the linearized IEEE 39-bus system in Section V. Finally, Appendices I, II, and III gather the proofs of our results.

Notation: We denote the integer interval from a to b, inclusive, with $[\![a, b]\!]$. For a set $\Lambda \subseteq \mathbb{C}$, we say that $Re(\Lambda) \leq 0$ (resp. $Re(\Lambda) = 0$) if the real part of each $\lambda \in \Lambda$ verifies $Re(\lambda) \leq 0$ (resp. $Re(\lambda) = 0$). The norm of a matrix A is $\|A\| := \sup\limits_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \max\limits_{\|x\|=1} \|Ax\|$, its image is $\mathrm{Im}(A)$, and the set of its eigenvalues is $\Lambda(A)$. If A is positive definite, denoted $A \succ 0$, then its extremal eigenvalues are λ_{min}^A and λ_{max}^A and A generates a vector norm $\|x\|_A := \sqrt{x^\top Ax}$. The controllability matrix of pair (A, B) is $\mathcal{C}(A, B) := \big[B\, AB\, \dots\, A^{n-1}B\big]$. For a matrix $B \in \mathbb{R}^{n \times m}$ and a set $\mathcal{U} \subseteq \mathbb{R}^m$, we denote the set $B\mathcal{U} := \big\{Bu : u \in \mathcal{U}\big\} \subseteq \mathbb{R}^n$. The block diagonal matrix composed of matrices $A_1, \dots, A_n$ is denoted by $\mathrm{diag}(A_1, \dots, A_n)$. The zero matrix of size $n \times m$ is denoted by $0_{n,m}$, the identity matrix of size n is I_n, and the vector of ones is $\mathbf{1}_n \in \mathbb{R}^n$. The convex hull of a set $\mathcal{Z}$ is denoted by $\mathrm{co}(\mathcal{Z})$, its dimension by $\dim(\mathcal{Z})$, its boundary by $\partial\mathcal{Z}$, its interior by $\mathrm{int}(\mathcal{Z})$, and its orthogonal complement by $\mathcal{Z}^\perp$. The set of time functions taking value in $\mathcal{Z}$ is denoted $\mathcal{F}(\mathcal{Z}) := \big\{f : [0, +\infty) \to \mathcal{Z}\big\}$. The Minkowski addition of sets $\mathcal{X}$ and $\mathcal{Y}$ in $\mathbb{R}^n$ is $\mathcal{X} \oplus \mathcal{Y} := \big\{x + y : x \in \mathcal{X}, y \in \mathcal{Y}\big\}$ and their Minkowski difference is $\mathcal{X} \ominus \mathcal{Y} := \big\{z \in \mathbb{R}^n : \{z\} \oplus \mathcal{Y} \subseteq \mathcal{X}\big\}$. The operator $\mathrm{span}(\cdot)$ maps a set of vectors to their linear span.

II. Networks preliminaries

In this section, we introduce the network under study and our two problems of interest. Inspired by [21], we consider a network of q linear subsystems of dynamics

$$\dot{x}_1(t) = A_1x_1(t) + \bar{B}_1\bar{u}_1(t) + \sum_{k \in \mathcal{N}_1} D_{1,k}x_k(t) \qquad \text{(1-1)}$$
$$\vdots$$
$$\dot{x}_q(t) = A_qx_q(t) + \bar{B}_q\bar{u}_q(t) + \sum_{k \in \mathcal{N}_q} D_{q,k}x_k(t), \qquad \text{(1-q)}$$

with initial states $x_i(0) = x_i^0 \in \mathbb{R}^{n_i}$ and *bounded* admissible control inputs $\bar{u}_i(t) \in \bar{\mathcal{U}}_i := [-1, 1]^{m_i}$ for $i \in [\![1, q]\!]$. The set of neighbors of subsystem i is denoted by $\mathcal{N}_i \subseteq [\![1, q]\!]$ with $i \notin \mathcal{N}_i$, while $A_i \in \mathbb{R}^{n_i \times n_i}$, $\bar{B}_i \in \mathbb{R}^{n_i \times m_i}$, and $D_{i,k} \in \mathbb{R}^{n_i \times n_k}$ are constant matrices. Let us now define our notion of finite-time *component stabilizability*.

Definition 1: Tuple $(A, \bar{B}, \bar{\mathcal{U}})$ is *stabilizable* if there exists a time $T \geq 0$ and an admissible control signal $\bar{u} \in \mathcal{F}(\bar{\mathcal{U}})$ driving the state of system $\dot{x}(t) = Ax(t) + \bar{B}\bar{u}(t)$ from any $x^0 \in \mathbb{R}^n$ to $x(T) = 0$.

Building on the component stabilizability of the subsystems (1-1) to (1-q), we will derive conditions on the stabilizability of the overall network. To do so, we define network state $X(t) := \big(x_1(t), x_2(t), \dots, x_q(t)\big) \in \mathbb{R}^{n_\Sigma}$ and control input $\bar{u}(t) := \big(\bar{u}_1(t), \dots, \bar{u}_q(t)\big) \in \bar{\mathcal{U}} := \bar{\mathcal{U}}_1 \times \ldots \times \bar{\mathcal{U}}_q \subseteq \mathbb{R}^{m_\Sigma}$ of dimensions $n_\Sigma := n_1 + \ldots + n_q$ and $m_\Sigma := m_1 + \ldots + m_q$, respectively. Network dynamics (1) can then be written more concisely as

$$\dot{X}(t) = (A + D)X(t) + \bar{B}\bar{u}(t), \qquad (2)$$

with initial state $X(0) = X_0 := \big(x_1^0, \dots, x_q^0\big) \in \mathbb{R}^{n_\Sigma}$ and constant matrices $A := \mathrm{diag}(A_1, \dots, A_q)$, $\bar{B} := \mathrm{diag}\big(\bar{B}_1, \dots, \bar{B}_q\big)$ and $D := \big(D_{i,j}\big)_{(i,j) \in [\![1,q]\!]}$ with $D_{i,k} = 0_{n_i,n_k}$ if $k \notin \mathcal{N}_i$. Since our objective is to investigate connected networks, we assume that $D \neq 0$.

Following, for instance, an adversarial cyber-attack [6], [11] of network (1), subsystem (1-q) suffers a loss of control authority over a number $p_q \in [\![1, m_q]\!]$ of its m_q actuators. We split the nominal input $\bar{u}_q$ between the remaining controlled inputs $u_q \in \mathcal{F}(\mathcal{U}_q), \mathcal{U}_q = [-1, 1]^{m_q - p_q}$ and the uncontrolled and possibly undesirable inputs $w_q \in \mathcal{F}(\mathcal{W}_q)$, $\mathcal{W}_q = [-1, 1]^{p_q}$. We accordingly split matrix $\bar{B}_q$ into $B_q \in \mathbb{R}^{n_q \times (m_q - p_q)}$ and $C_q \in \mathbb{R}^{n_q \times p_q}$, so that the dynamics of subsystem (1-q) become

$$\dot{x}_q(t) = A_qx_q(t) + B_qu_q(t) + C_qw_q(t) + \sum_{k \in \mathcal{N}_q} D_{q,k}x_k(t), \qquad (3)$$

with unchanged initial state $x_q(0) = x_q^0 \in \mathbb{R}^{n_q}$. We adopt the resilience framework of [7], [10] where controller $u_q(t)$ has real-time knowledge of the undesirable inputs $w_q(t)$ thanks to sensors located on each actuator. This assumption of real-time knowledge was relaxed in [22] by considering a controller inflicted by a constant actuation delay. Beyond this additional layer of complexity, the resilience conditions were extremely similar to those with immediate knowledge of the perturbations, which is why we make this simplifying assumption.

Our central objective is to study how the partial loss of control authority over actuators of subsystem (1-q) affects the *stabilizability* of the whole network. To adapt this property to malfunctioning system (3), we first need the notion of *resilient reachability* introduced in [7].

Definition 2: A target $x_{goal} \in \mathbb{R}^n$ is *resiliently reachable* from $x^0 \in \mathbb{R}^n$ by malfunctioning system $\dot{x}(t) = Ax(t) + Bu(t) + Cw(t)$ if for all $w \in \mathcal{F}(\mathcal{W})$, there exists $T \geq 0$ and $u \in \mathcal{F}(\mathcal{U})$ such that $u(t)$ only depends on $w(t)$ and the solution exists, is unique, and $x(T) = x_{goal}$.

Definition 3: Tuple $(A, B, C, \mathcal{U}, \mathcal{W})$ is *resiliently stabilizable* if $0 \in \mathbb{R}^n$ is resiliently reachable from any $x^0 \in \mathbb{R}^n$ by malfunctioning system $\dot{x}(t) = Ax(t) + Bu(t) + Cw(t)$.

Network dynamics (2) are also impacted by the loss of control authority in subsystem (1-q). We define the network control input $u(t) := \big(\bar{u}_1(t), \ldots, \bar{u}_{q-1}(t), u_q(t)\big) \in \mathcal{U} := \bar{\mathcal{U}}_1 \times \ldots \times \bar{\mathcal{U}}_{q-1} \times \mathcal{U}_q \subseteq \mathbb{R}^{m_\Sigma - p_q}$. Network dynamics (2) then become

$$\dot{X}(t) = (A + D)X(t) + Bu(t) + Cw_q(t), \tag{4}$$

with unchanged initial state $X(0) = X_0 \in \mathbb{R}^{n_\Sigma}$ and constant matrices $B = \operatorname{diag}\big(\bar{B}_1, \ldots, \bar{B}_{q-1}, B_q\big)$, $C = \begin{pmatrix} 0_{n_\Sigma - n_q, p_q} \\ C_q \end{pmatrix}$.

Definition 4: Network (4) is *resiliently stabilizable* if tuple $\big(A + D, B, C, \mathcal{U}, \mathcal{W}_q\big)$ is resiliently stabilizable.

We are now led to the following problem of interest.

Problem 1: Assuming that tuple $(A_q, B_q, C_q, \mathcal{U}_q, \mathcal{W}_q)$ is resiliently stabilizable and tuples $(A_i, \bar{B}_i, \bar{\mathcal{U}}_i)$ are stabilizable for $i \in [\![1, q-1]\!]$, under what conditions network (4) is resiliently stabilizable?

Note that the resilience framework for network (4) allows its control input $u(t)$ to depend on undesirable input $w_q(t)$, which presupposes that all subsystems are aware of the attack.

Remark 1 (Extension to non-zero targets): If instead of resilient stabilizability, we want to resiliently drive the state of network (4) to any non-zero target, we need to use the notion of *controllability* instead of stabilizability. As highlighted in work [10], this change of objective barely modifies resilience conditions and hence we only treat this case in the extended version of this work[1].

After investigating the ideal case of Problem 1 where tuple $(A_q, B_q, C_q, \mathcal{U}_q, \mathcal{W}_q)$ is resiliently stabilizable, we will consider the more problematic scenario where it is not resilient and study whether the rest of the network remain stabilizable despite the perturbations arising from the coupling with malfunctioning subsystem (3). Let $\chi(t)$ be the combined state of all other subsystems, i.e., $\chi(t) := \big(x_1(t), \ldots, x_{q-1}(t)\big)$ Then,

$$\dot{\chi}(t) = \hat{A}\chi(t) + \hat{B}\hat{u}(t) + \hat{D}\chi(t) + D_{\text{-},q}x_q(t), \tag{5}$$

with $\chi_0 := \big(x_1^0, \ldots, x_{q-1}^0\big)$, $\hat{A} := \operatorname{diag}\big(A_1, \ldots, A_{q-1}\big)$, $\hat{B} := \operatorname{diag}\big(\bar{B}_1, \ldots, \bar{B}_{q-1}\big)$, and $\hat{u}(t) := \big(\bar{u}_1(t), \ldots, \bar{u}_{q-1}(t)\big) \in \hat{\mathcal{U}} := \bar{\mathcal{U}}_1 \times \ldots \times \bar{\mathcal{U}}_{q-1} = [-1, 1]^{m_\Sigma - m_q}$. We have also split matrix D such that $D =$

$$\left[\begin{array}{ccc|c} 0_{n_1,n_1} & \ldots & D_{1,q\text{-}1} & D_{1,q} \\ \vdots & \ddots & \vdots & \vdots \\ D_{q\text{-}1,1} & \ldots & 0_{n_{q\text{-}1},n_{q\text{-}1}} & D_{q\text{-}1,q} \\ \hline D_{q,1} & \ldots & D_{q,q\text{-}1} & 0_{n_q,n_q} \end{array}\right] := \left[\begin{array}{c|c} \hat{D} & D_{\text{-},q} \\ \hline D_{q,\text{-}} & 0_{n_q,n_q} \end{array}\right].$$

Definition 5: System (5) is *resiliently stabilizable* if for every $X_0 \in \mathbb{R}^{n_\Sigma}$ and every $w_q \in \mathcal{F}(\mathcal{W}_q)$ there exists $T \geq 0$ and $u \in \mathcal{F}(\mathcal{U})$ such that the solution to the *entire network* (4) exists, is unique, and $\chi(T) = 0$.

The resilient stabilizability of subsystem (5) depends on the initial state X_0 of the entire network (4) and on the undesirable input w_q perturbing state χ through the coupling term $D_{\text{-},q}x_q$ in (5). If stabilizing state χ is impossible, the next best objective would be to maintain it around the origin.

[1] https://arxiv.org/abs/2306.16588

Definition 6: System (5) is *resiliently bounded* if for every $X_0 \in \mathbb{R}^{n_\Sigma}$ and every $w_q \in \mathcal{F}(\mathcal{W}_q)$ there exists $b \geq 0$ and $u \in \mathcal{F}(\mathcal{U})$ such that the solution to the *entire network* (4) exists, is unique, and $\|\chi(t)\| \leq b$ for all $t \geq 0$.

We can then state our second problem of interest.

Problem 2: Assuming that $(A_q, B_q, C_q, \mathcal{U}_q, \mathcal{W}_q)$ is *not* resiliently stabilizable and $(A_i, \bar{B}_i, \bar{\mathcal{U}}_i)$ is stabilizable for $i \in [\![1, q-1]\!]$, under what conditions system (5) is resiliently stabilizable or resiliently bounded ?

Note that Problem 2 does not try to resiliently stabilize subsystem (3) along with the other subsystems. Indeed, the only way to do so would rely on the coupling term $\sum D_{q,k}x_k$, which is going to 0 as the other subsystems are getting stabilized. Therefore, malfunctioning network (4) is not resiliently stabilizable when tuple $(A_q, B_q, C_q, \mathcal{U}_q, \mathcal{W}_q)$ is not resiliently stabilizable. We start by investigating Problem 1.

III. Stabilizability of Resilient Networks

In this section, we build on several background results from stabilizability and resilience theories to tackle Problem 1.

A. Stabilizability results

Malfunctioning network (4) can only be resiliently stabilizable if the network was stabilizable before the malfunction. We then start by investigating the finite-time stabilizability of initial network (2). Since $0 \in \operatorname{int}(\bar{\mathcal{U}})$, we can use Sontag's stabilizability condition for systems with *bounded* inputs [23].

Theorem 1: Network (2) is stabilizable in finite-time if and only if $\operatorname{rank}\mathcal{C}\big(A + D, \bar{B}\big) = n_\Sigma$ and $Re\big(\Lambda(A + D)\big) \leq 0$.

Since Problem 1 aims at relating the resilient stabilizability of malfunctioning network (4) to that of its subsystems, a preliminary step in this direction is to relate the stabilizability of initial network (2) to that of its subsystems. To do so, we introduce $\mu_{\bar{B}}$ inspired from the *distance to uncontrollability* of [24] as

$$\mu_{\bar{B}}(A) := \min_{D_1 \in \mathbb{R}^{n \times n}} \big\{\|D_1\| : \operatorname{rank}\mathcal{C}\big(A + D_1, \bar{B}\big) < n_\Sigma\big\}.$$

Trivially, $\|D\| < \mu_{\bar{B}}(A)$ is sufficient to satisfy the rank condition of Theorem 1.

To relate the eigenvalue condition of Theorem 1 to the stability of all pairs $(A_i, \bar{B}_i)$, we introduce the *real stability radius* of A from [25]:

$$r_{\mathbb{R}}(A) := \inf_{D_1 \in \mathbb{R}^{n \times n}} \big\{\|D_1\| : \max\big\{Re\big(\Lambda(A + D_1)\big)\big\} > 0\big\}.$$

Trivially, $\|D\| < r_{\mathbb{R}}(A)$ is sufficient to satisfy the eigenvalue condition of Theorem 1. We can now relate the stabilizability of network (2) to the stabilizability of its subsystems.

Proposition 1: If $\|D\| < \min\{\mu_{\bar{B}}(A), r_{\mathbb{R}}(A)\}$, then network (2) and all tuples $(A_i, \bar{B}_i, \bar{\mathcal{U}}_i)$ are stabilizable in finite-time for $i \in [\![1, q]\!]$.

Proof: The finite-time stabilizability of network (2) follows directly from Theorem 1 and the definitions of $\mu_{\bar{B}}(A)$ and $r_{\mathbb{R}}(A)$.

To obtain the stabilizability of tuples $(A_i, \bar{B}_i, \bar{\mathcal{U}}_i)$, we recall that A and $\bar{B}$ are block diagonal matrices of A_i and $\bar{B}_i$ respectively. Since $D \neq 0$, the assumption $\|D\| < \mu_{\bar{B}}(A)$ yields

$0 < \mu_{\bar{B}}(A)$, i.e., $\operatorname{rank} \mathcal{C}(A, \bar{B}) = n_\Sigma$. Then, $\operatorname{rank} \mathcal{C}(A, \bar{B}) = n_\Sigma$ is equivalent to $\operatorname{rank} \mathcal{C}(A_i, \bar{B}_i) = n_i$ for all $i \in [\![1, q]\!]$.

Similarly, $D \neq 0$ and $\|D\| < r_{\mathbb{R}}(A)$ yield $0 < r_{\mathbb{R}}(A)$, i.e., $Re\big(\Lambda(A)\big) \leq 0$. Since A is a block diagonal matrix of A_i, we have $Re\big(\Lambda(A_i)\big) \leq 0$ for all $i \in [\![1, q]\!]$.

Then, applying Theorem 1 to $\dot{x}_i(t) = A_i x_i(t) + \bar{B}_i \bar{u}_i(t)$ yields the finite-time stabilizability of tuple $(A_i, \bar{B}_i, \bar{\mathcal{U}}_i)$. ■

Now that we have stabilizability conditions for initial network (2), we can investigate its resilient stabilizability after a partial loss of control authority.

B. Resilient stabilizability results

To address Problem 1, we investigate the case where tuple $(A_q, B_q, C_q, \mathcal{U}_q, \mathcal{W}_q)$ is resiliently stabilizable following Definition 3. Resilience conditions established in [10] rely on Hájek's differential games approach [26]. These conditions consider the following dynamics associated to malfunctioning system (3):

$$\dot{x}_q(t) = A_q x_q(t) + z_q(t), \quad z_q(t) \in \mathcal{Z}_q, \tag{6}$$

with unchanged initial state $x_q(0) = x_q^0 \in \mathbb{R}^{n_q}$ and *resilient control set* $\mathcal{Z}_q \subseteq \mathbb{R}^{n_q}$. Set $\mathcal{Z}_q$ results from the Minkowski difference between the set of admissible control inputs $B_q\mathcal{U}_q := \big\{B_q u_q : u_q \in \mathcal{U}_q\big\}$ and the opposite of the set of undesirable inputs $C_q\mathcal{W}_q := \big\{C_q w_q : w_q \in \mathcal{W}_q\big\}$, i.e.,

$$\begin{aligned}
\mathcal{Z}_q &:= \big(B_q\mathcal{U}_q \ominus (-C_q\mathcal{W}_q)\big) \cap B_q\mathcal{U}_q \\
&= \big\{z_q \in B_q\mathcal{U}_q : \{z_q\} \oplus (-C_q\mathcal{W}_q) \subseteq B_q\mathcal{U}_q\big\} \\
&= \big\{z_q \in B_q\mathcal{U}_q : z_q - C_q w_q \in B_q\mathcal{U}_q \text{ for all } w_q \in \mathcal{W}_q\big\}.
\end{aligned}$$

Informally, $\mathcal{Z}_q$ represents the remaining control available after counteracting any undesirable input. As stated in [10], Hájek's duality result of [26] establishes the equivalence between the stabilizability of system (6) and the resilient stabilizability of tuple $(A_q, B_q, C_q, \mathcal{U}_q, \mathcal{W}_q)$.

Similarly to the approach of [10], we study the resilience of malfunctioning network (4) by constructing its resilient control set $\mathcal{Z} := \big[B\mathcal{U} \ominus (-C\mathcal{W}_q)\big] \cap B\mathcal{U} \subseteq \mathbb{R}^{n_\Sigma}$. We can then state the first resilience condition of [10].

Proposition 2 (Sufficient condition [10]): If $\operatorname{int}(\mathcal{Z}) \neq \emptyset$, then network (4) is resiliently stabilizable if and only if $Re\big(\Lambda(A + D)\big) \leq 0$.

The main issue with Proposition 2 is the restrictive requirement that $\mathcal{Z}$ has a non-empty interior in $\mathbb{R}^{n_\Sigma}$. To better understand what this requirement entails, we investigate the structure of set $\mathcal{Z}$ in the following result.

Proposition 3: The resilient control set of network (4) is the Cartesian product of the input sets of its subsystems: $\mathcal{Z} = \bar{B}_1\bar{\mathcal{U}}_1 \times \ldots \times \bar{B}_{q-1}\bar{\mathcal{U}}_{q-1} \times \mathcal{Z}_q$.

Proof: We prove this equality by showing both inclusions.

Take $z = (z_1, \ldots, z_q) \in \mathcal{Z}$. We want to show that $z_i \in \bar{B}_i\bar{\mathcal{U}}_i$ for $i \in [\![1, q-1]\!]$ and that $z_q \in \mathcal{Z}_q$. Let w_q be any element in $\mathcal{W}_q$. Since $z \in B\mathcal{U} \ominus (-C\mathcal{W}_q)$, $z - Cw_q \in B\mathcal{U}$. Additionally, since $\mathcal{U} = \bar{\mathcal{U}}_1 \times \ldots \times \bar{\mathcal{U}}_{q-1} \times \mathcal{U}_q$, there exists $u = (\bar{u}_1, \ldots, \bar{u}_{q-1}, u_q) \in \mathcal{U}$ such that

$$z - Cw_q = \begin{bmatrix} z_1 \\ \vdots \\ z_{q-1} \\ z_q - C_q w_q \end{bmatrix} = Bu = \begin{bmatrix} \bar{B}_1\bar{u}_1 \\ \vdots \\ \bar{B}_{q-1}\bar{u}_{q-1} \\ B_q u_q \end{bmatrix}.$$

Then, $z_i \in \bar{B}_i\bar{\mathcal{U}}_i$ for $i \in [\![1, q-1]\!]$ and for all $w_q \in \mathcal{W}_q$ we have $z_q - C_q w_q \in B_q\mathcal{U}_q$, i.e., $z_q \in \mathcal{Z}_q$. Thus, $\mathcal{Z} \subseteq \Pi_{i=1}^{q-1}\bar{B}_i\bar{\mathcal{U}}_i \times \mathcal{Z}_q$.

On the other hand, let $\bar{u}_i \in \bar{\mathcal{U}}_i$ for $i \in [\![1, q-1]\!]$, $z_q \in \mathcal{Z}_q$ and define $z = \big(\bar{B}_1\bar{u}_1, \ldots, \bar{B}_{q-1}\bar{u}_{q-1}, z_q\big)$. We want to show that $z \in \mathcal{Z}$. Let $w_q \in \mathcal{W}_q$. Since $z_q \in \mathcal{Z}_q$, there exists $u_q \in \mathcal{U}_q$ such that $z_q - C_q w_q = B_q u_q$. Then,

$$z - Cw_q = \begin{bmatrix} \bar{B}_1\bar{u}_1 \\ \vdots \\ \bar{B}_{q-1}\bar{u}_{q-1} \\ z_q \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ C_q \end{bmatrix} w_q = \begin{bmatrix} \bar{B}_1\bar{u}_1 \\ \vdots \\ \bar{B}_{q-1}\bar{u}_{q-1} \\ B_q u_q \end{bmatrix},$$

i.e., $z - Cw_q \in B\mathcal{U}$. Thus, $z \in \mathcal{Z}$ and $\Pi_{i=1}^{q-1}\bar{B}_i\bar{\mathcal{U}}_i \times \mathcal{Z}_q \subseteq \mathcal{Z}$. ■

Now that we have a better understanding of the structure of set $\mathcal{Z}$, we can reformulate Proposition 2.

Proposition 4: If $\operatorname{rank}(\bar{B}_i) = n_i$ for all $i \in [\![1, q-1]\!]$, $\operatorname{int}(\mathcal{Z}_q) \neq \emptyset$ and $\|D\| < r_{\mathbb{R}}(A)$, then network (4) is resiliently stabilizable.

Proof: Since $\operatorname{rank}(\bar{B}_i) = n_i$ and $\bar{\mathcal{U}}_i = [-1, 1]^{m_i}$ with $\bar{B}_i \in \mathbb{R}^{n_i \times m_i}$, we have $\operatorname{int}(\bar{B}_i\bar{\mathcal{U}}_i) \neq \emptyset$. Then, according to Proposition 3, $\operatorname{int}(\mathcal{Z}) \neq \emptyset$. By assumption, we have $\|D\| < r_{\mathbb{R}}(A)$, i.e., $Re(\Lambda(A + D)) \leq 0$. Then, Proposition 2 states that network (4) is resiliently stabilizable. ■

Proposition 4 provides a straightforward resilient stabilizability condition for network (4). However, both Propositions 2 and 4 require all control matrices to be full rank, which is not necessary for stabilizability. To remove this restrictive requirement, work [10] relied on a matrix $Z \in \mathbb{R}^{n_\Sigma \times r}$ with $r := \dim(\mathcal{Z})$ such that $\operatorname{Im}(Z) = \operatorname{span}(\mathcal{Z})$. In practice, matrix Z is built by collating r linearly independent vectors from set $\mathcal{Z}$. We can now state the equivalence condition from [10].

Theorem 2 (Equivalence condition [10]): Network (4) is resiliently stabilizable if and only if $Re\big(\Lambda(A + D)\big) \leq 0$, $\operatorname{rank}\big(\mathcal{C}(A + D, Z)\big) = n_\Sigma$ and there is no real eigenvector v of $(A + D)^\top$ satisfying $v^\top z \leq 0$ for all $z \in \mathcal{Z}$.

Following Problem 1, we want to relate the resilient stabilizability of network (4) to the stabilizability of its subsystems. Hence, we need to isolate the role played by coupling matrix D in the resilient stabilizability of network (4).

Corollary 1: If $\|D\| < \min\big\{r_{\mathbb{R}}(A), \mu_Z(A)\big\}$ and there is no real eigenvector v of $(A + D)^\top$ satisfying $v^\top z \leq 0$ for all $z \in \mathcal{Z}$, then network (4) is resiliently stabilizable.

Proof: Assumption $\|D\| < \min\{r_{\mathbb{R}}(A), \mu_Z(A)\}$ satisfies the eigenvalue and rank conditions of Theorem 2 which yields the result. ■

When $\mathcal{Z}$ is not of full dimension, the eigenvector condition of Corollary 1 is difficult to verify. Indeed, the space $\mathcal{Z}^\perp$ is non-trivial and thus might encompass a real eigenvector of $A + D$ even if none of the eigenvectors of A are part of $\mathcal{Z}^\perp$. Intuitively, when D is small, the eigenvectors of $A + D$ should be 'close' to those of A. This intuition is formalized in Corollary 7.2.6 of [27], but the complexity of its statement prevents the derivation of a simple condition to be verified by A and D. Thus, we choose to remain with Corollary 1 and Proposition 4 as solutions to Problem 1.

IV. Stabilizability of nonresilient networks

In this section, we address Problem 2 by studying the network-wide repercussions resulting from the partial loss of control authority in nonresilient subsystem (3).

We are interested in the eventuality where tuple $(A_q, B_q, C_q, \mathcal{U}_q, \mathcal{W}_q)$ is not resiliently stabilizable and more specifically to the case $-C_q\mathcal{W}_q \not\subseteq B_q\mathcal{U}_q$, i.e., $\mathcal{Z}_q = \emptyset$. This condition means that subsystem (3) lost control over actuators whose combined actions cannot be counteracted by the remaining controlled actuators of subsystem (3). In other words, some undesirable actions $C_q w_q$ cannot be canceled by any admissible control $B_q u_q$, which prevents the resilient stabilizability of subsystem q, as demonstrated in Lemma 6 of [10].

To evaluate the resilient stabilizability of network (4), we need to study the worst-case scenario where w_q is the most destabilizing undesirable input for subsystem (3). If A_q is not Hurwitz, these destabilizing inputs w_q can drive the state x_q to infinity. In this situation, coupling terms $D_{i,q}x_q$ impacting subsystems (1-i) can become unbounded preventing to stabilize these other subsystems. We will then focus on the case where A_q is Hurwitz, so that the state x_q cannot be forced to diverge by w_q. Then, the term $D_{\text{-},q}x_q$ perturbing subsystem (5) is bounded and might be counteracted if controller $\hat{B}\hat{u}$ is strong enough.

To address Problem 2, we will quantify the maximal degree of non-resilience of subsystem (3) despite which subsystem (5) remain resiliently stabilizable in the sense of Definition 5. We start by calculating how far can w_q force state x_q despite the best u_q and the Hurwitzness of A_q.

Proposition 5: If A_q is Hurwitz and $-C_q\mathcal{W}_q \not\subseteq B_q\mathcal{U}_q$, then for all $t \geq 0$ the following holds:

$$\|x_q(t)\|_{P_q} \leq e^{-\alpha_q t}\left(\big\|x_q^0\big\|_{P_q} + \int_0^t e^{\alpha_q \tau}\beta_q(\tau)\, d\tau\right), \tag{7}$$

for all $P_q = P_q^\top \succ 0$, $Q_q \succ 0$ such that $A_q^\top P_q + P_q A_q = -Q_q$ and with $\beta_q(\tau) := z_{max}^{P_q} + \|D_{q,\text{-}}\chi(\tau)\|_{P_q}$,

$$\alpha_q := \frac{\lambda_{min}^{Q_q}}{2\lambda_{max}^{P_q}}, \quad z_{max}^{P_q} := \max_{w_q \in \mathcal{W}_q} \min_{u_q \in \mathcal{U}_q} \big\|C_q w_q + B_q u_q\big\|_{P_q}.$$

Proof: Since A_q is Hurwitz, there exist matrices $P_q \succ 0$ and $Q_q \succ 0$ such that $A_q^\top P_q + P_q A_q = -Q_q$ according to Lyapunov theory [28]. Let us consider any such pair (P_q, Q_q). Then, inspired by Example 15 of [28], we study the P_q-norm of x_q, i.e., $x_q^\top P_q x_q = \|x_q\|_{P_q}^2$ when state x_q is following dynamics (3). We obtain

$$\begin{aligned}\frac{d}{dt}\|x_q(t)\|_{P_q}^2 &= \dot{x}_q^\top P_q x_q + x_q^\top P_q \dot{x}_q \\ &= -x_q^\top Q_q x_q + 2x_q^\top P_q \left(B_q u_q + C_q w_q + \sum_{i=1}^{q-1} D_{q,i}x_i\right).\end{aligned}$$

Note that $\sum_{i=1}^{q-1} D_{q,i}x_i = D_{q,\text{-}}\chi$. Since $P_q \succ 0$, the Cauchy-Schwarz inequality [27] yields

$$\begin{aligned}x_q^\top P_q D_{q,\text{-}}\chi &\leq \|x_q\|_{P_q}\|D_{q,\text{-}}\chi\|_{P_q} \\ x_q^\top P_q(B_q u_q + C_q w_q) &\leq \|x_q\|_{P_q}\|B_q u_q + C_q w_q\|_{P_q}.\end{aligned}$$

We will demonstrate the stabilizing property of the control u_q minimizing $\|B_q u_q + C_q w_q\|_{P_q}$ when w_q is chosen to maximize this norm. By definition, these choices of u_q and w_q yield $\|B_q u_q + C_q w_q\|_{P_q} \leq z_{max}^{P_q}$. Then,

$$\frac{d}{dt}\|x_q\|_{P_q}^2 \leq -x_q^\top Q_q x_q + 2\|x_q\|_{P_q}\left(z_{max}^{P_q} + \|D_{q,\text{-}}\chi\|_{P_q}\right).$$

Since $Q_q \succ 0$, we have $-x_q^\top Q_q x_q \leq -\lambda_{min}^{Q_q} x_q^\top x_q$ [29] and $\|x_q\|_{P_q}^2 \leq \lambda_{max}^{P_q} x_q^\top x_q$ leads to $-x_q^\top x_q \leq \frac{-1}{\lambda_{max}^{P_q}}\|x_q\|_{P_q}^2$. Hence, we obtain

$$\frac{d}{dt}\|x_q\|_{P_q}^2 \leq -2\alpha_q\|x_q(t)\|_{P_q}^2 + 2\beta_q(t)\|x_q(t)\|_{P_q},$$

by definition of α_q and β_q. We introduce $y_q(t) := \|x_q(t)\|_{P_q}$, so that we have

$$\frac{d}{dt}y_q^2(t) = 2y_q(t)\dot{y}_q(t) \leq -2\alpha_q y_q(t)^2 + 2\beta_q(t)y_q(t).$$

For $y_q(t) > 0$, we then have $\dot{y}_q(t) \leq -\alpha_q y_q(t) + \beta_q(t)$.

Let $f_q(t,s) := -\alpha_q s + \beta_q(t)$. The solution of the differential equation $\dot{s}(t) = f_q\big(t, s(t)\big)$, $s(0) = \|x_q^0\|_{P_q}$ is $s(t) = e^{-\alpha_q t}\left(\|x_q^0\|_{P_q} + \int_0^t e^{\alpha_q\tau}\beta_q(\tau)\,d\tau\right)$. Since $f_q(t,s)$ is Lipschitz in s and continuous in t, $\dot{y}_q(t) \leq f_q\big(t, y_q(t)\big)$ and $y_q(0) = z(0)$, the Comparison Lemma of [29] states that $y_q(t) \leq s(t)$ for all $t \geq 0$, hence (7) holds. ■

Note that the definition of $z_{max}^{P_q}$ in Proposition 5 implies that w_q is chosen first and the controller u_q reacts optimally to it. The objective function not being concave-convex, there is an information imbalance giving an advantage to the second player. If we wanted instead the undesirable input to react optimally to any controller, we could use $z' := \min\limits_{u_q \in \mathcal{U}_q} \max\limits_{w_q \in \mathcal{W}_q} \big\|C_q w_q + B_q u_q\big\|_{P_q}$ in place of $z_{max}^{P_q}$. Note that $z' \geq z_{max}^{P_q}$.

Now that we have bounded x_q, we can evaluate how term $D_{\text{-},q}x_q(t)$ impacts the network state $\chi(t)$ by building on Proposition 5 and reusing P_q, Q_q, α_q, β_q and $z_{max}^{P_q}$. We will first investigate the scenario where $\hat{B}$ is full rank before requiring only controllability of pair $\big(\hat{A} + \hat{D}, \hat{B}\big)$.

A. Fully-actuated networks

In this section we assume that the combined control matrix of the first $q-1$ subsystems, i.e., $\hat{B}$ is full rank.

Proposition 6: If $\hat{A} + \hat{D}$ and A_q are Hurwitz, $\hat{B}$ is full rank, and $C_q\mathcal{W}_q \not\subseteq B_q\mathcal{U}_q$, then for any $\hat{P} \succ 0$ and $\hat{Q} \succ 0$ such that $(\hat{A} + \hat{D})^\top \hat{P} + \hat{P}(\hat{A} + \hat{D}) = -\hat{Q}$ we introduce the positive constants $b_{min}^{\hat{P}} := \min\limits_{\hat{u} \in \partial\hat{\mathcal{U}}}\big\{\|\hat{B}\hat{u}\|_{\hat{P}}\big\}$, $\alpha := \frac{\lambda_{min}^{\hat{Q}}}{2\lambda_{max}^{\hat{P}}}$, $\gamma := \sqrt{\frac{\max\Lambda(D_{\text{-},q}^\top \hat{P} D_{\text{-},q})}{\lambda_{min}^{P_q}}}$ and $\gamma_q := \sqrt{\frac{\max\Lambda(D_{q,\text{-}}^\top P_q D_{q,\text{-}})}{\lambda_{min}^{\hat{P}}}}$.

If $\alpha\alpha_q \neq \gamma\gamma_q$, then there exist $h_\pm \in \mathbb{R}$ and $r_\pm \in \mathbb{R}$ such that

$$\|\chi(t)\|_{\hat{P}} \leq \max\left\{0,\ \frac{\gamma z_{max}^{P_q} - \alpha_q b_{min}^{\hat{P}}}{\alpha\alpha_q - \gamma\gamma_q} + \sum_{*\in\{+,-\}} h_* e^{(r_*-\alpha_q)t}\right\}. \tag{8}$$

If $\alpha\alpha_q = \gamma\gamma_q$, there are constants $h_\pm \in \mathbb{R}$ such that

$$\|\chi(t)\|_{\hat{P}} \leq \max\left\{0,\ \frac{\gamma z_{max}^{P_q} - \alpha_q b_{min}^{\hat{P}}}{\alpha + \alpha_q} t + h_+ + h_- e^{-(\alpha+\alpha_q)t}\right\}. \tag{9}$$

Proof: The proof is given in Appendix I ∎

We can now derive conditions for subsystem (5) to be resiliently stabilizable despite the perturbations created by x_q. These conditions solve Problem 2 in the fully-actuated network scenario.

Theorem 3: If $\hat{A} + \hat{D}$ and A_q are Hurwitz, $\hat{B}$ is full rank, $C_q\mathcal{W}_q \not\subseteq B_q\mathcal{U}_q$, $\gamma\gamma_q \leq \alpha\alpha_q$ and $\gamma z_{max}^{P_q} < \alpha_q b_{min}^{\hat{P}}$, then subsystem (5) is resiliently stabilizable in finite time.

Proof: Let us first consider the case $\gamma\gamma_q = \alpha\alpha_q$. Since $\alpha > 0$ and $\alpha_q > 0$, the exponential term in (9) goes to zero asymptotically. By assumption $\gamma z_{max}^{P_q} - \alpha_q b_{min}^{\hat{P}} < 0$ and $\alpha + \alpha_q > 0$, so the ratio of these factors is negative. Because this ratio is multiplied by t in (9), there exists some time $T \geq 0$ such that for all $t \geq T$

$$\frac{\gamma z_{max}^{P_q} - \alpha_q b_{min}^{\hat{P}}}{\alpha + \alpha_q} t + h_+ + h_- e^{-(\alpha+\alpha_q)t} \leq 0.$$

Therefore, according to (9), subsystem (5) is resiliently stabilizable in finite time.

Now consider the case $\gamma\gamma_q < \alpha\alpha_q$. Using (19), we can easily show that this inequality is equivalent to $r_+ - \alpha_q < 0$. Since $r_- \leq r_+$, we also have $r_- - \alpha_q < 0$, so both exponential terms in (8) converge to zero. Additionally, the fraction term in (8) is negative, so the right-hand side of (8) reaches zero in finite time. Therefore, subsystem (5) is resiliently stabilizable in finite time. ∎

Let us now give some intuition concerning Theorem 3. Since γ is proportional to the norm of the matrix $D_{\cdot,q}$ which multiplies x_q in (5), γ quantifies the impact of nonresilient subsystem (3) of state x_q on the rest of the network (5) of state χ. Reciprocally, γ_q quantifies the impact of $\chi(t)$ on $x_q(t)$. On the other hand, $\alpha = \frac{\lambda_{min}^{\hat{Q}}}{2\lambda_{max}^{\hat{P}}}$ relates to the joint stability of the first $q-1$ subsystems of network (5), while α_q relates to the stability of malfunctioning subsystem (3). Therefore, condition $\gamma\gamma_q \leq \alpha\alpha_q$ follows the intuition that the magnitude of the perturbations arising from the coupling between subsystems (5) and (3) must be weaker than the stability of each of these subsystems.

We will now discuss the other stabilizability condition of Theorem 3, namely, $\gamma z_{max}^{P_q} < \alpha_q b_{min}^{\hat{P}}$. Since $z_{max}^{P_q}$ describes the magnitude of the destabilizing inputs in subsystem (3), term $\gamma z_{max}^{P_q}$ quantifies the destabilizing influence of w_q on the state of the rest of the network χ. On the other hand, $b_{min}^{\hat{P}}$ relates to the magnitude of the stabilizing inputs in subsystem (5) and α_q relates to the Hurwitzness of malfunctioning subsystem (3). Therefore, condition $\gamma z_{max}^{P_q} < \alpha_q b_{min}^{\hat{P}}$ carries the intuition that the stabilizing terms of the network must overcome the destabilizing ones.

Theorem 3 can also be used in an adversarial fashion, by identifying subsystems of the network which are not guaranteed to be resiliently stabilizable by Theorem 3.

Building on the bound of state χ from Proposition 6, we can now derive a closed-form bound on state x_q of the malfunctioning subsystem q. Indeed, the bound on $x_q(t)$ derived in Proposition 5 depends on $\chi(t)$ through the term $\beta_q(t)$.

Proposition 7: If $\hat{A}+\hat{D}$ and A_q are Hurwitz, $\hat{B}$ is full rank, $C_q\mathcal{W}_q \not\subseteq B_q\mathcal{U}_q$, and $\alpha\alpha_q \neq \gamma\gamma_q$, we can bound the state of subsystem (3) as

$$\|x_q(t)\|_{P_q} \leq \begin{cases} \max\{0,\ e^{-\alpha_q t}\|x_q^0\|_{P_q} + \delta(t)\} & \text{if } \|\chi(t)\|_{\hat{P}} > 0 \\ \frac{z_{max}^{P_q}}{\alpha_q} + \left(\|x_q^0\|_{P_q} - \frac{z_{max}^{P_q}}{\alpha_q}\right) e^{-\alpha_q t} & \text{otherwise,} \end{cases} \tag{10}$$

with
$$\delta(t) = \frac{\alpha z_{max}^{P_q} - \gamma_q b_{min}^{\hat{P}}}{\alpha\alpha_q - \gamma\gamma_q}\left(1 - e^{-\alpha_q t}\right) + e^{-\alpha_q t}\left(\left(e^{r_+ t} - 1\right)\frac{\gamma_q h_+}{r_+} + \left(e^{r_- t} - 1\right)\frac{\gamma_q h_-}{r_-}\right).$$

Proof: The proof is given in Appendix II ∎

The singular case $\alpha\alpha_q = \gamma\gamma_q$ is investigated in the ArXiv version of this paper[1].

Remark 2: The switch between bounds (10) is likely to be discontinuous. Indeed, the bound in (10) valid for $\|\chi(t)\|_{\hat{P}} > 0$ relies on all the overapproximations of bound (8), whereas the case $\|\chi(t)\|_{\hat{P}} = 0$ is derived without these overapproximations.

Thanks to Propositions 6 and 7, we now have a complete description of the network state after a nonresilient loss of control authority. These results relied on the full rank assumption of $\hat{B}$. Because this assumption might be too restrictive, we will now employ a different approach to bound the states of an underactuated network.

B. Underactuated networks

In this section, we will only assume that pair $(\hat{A}+\hat{D}, \hat{B})$ is controllable. Instead of the stabilizing control input of constant magnitude $\hat{B}\hat{u}(t) = -\frac{\chi(t)}{\|\chi(t)\|_{\hat{P}}} b_{min}^{\hat{P}}$ used in Proposition 6, we will employ a linear control to bound network state χ.

The controllability assumption leads to the existence of a matrix K such that $\hat{A} + \hat{D} - \hat{B}K$ is Hurwitz [29]. Then, for any $\hat{P} \succ 0$ and $\hat{Q} \succ 0$ such that $(\hat{A} + \hat{D} - \hat{B}K)^\top \hat{P} + \hat{P}(\hat{A} + \hat{D} - \hat{B}K) = -\hat{Q}$ [28], we can define the same constants α, γ, γ_q and $r_\pm$ as in Proposition 6.

Proposition 8: If pair $(\hat{A}+\hat{D}, \hat{B})$ is controllable, A_q is Hurwitz, $C_q\mathcal{W}_q \not\subseteq B_q\mathcal{U}_q$, $\gamma\gamma_q < \alpha\alpha_q$ and $\sup_{t\geq 0} b(t) \leq \frac{\sqrt{\lambda_{min}^{\hat{P}}}}{\|K\|}$, then system (5) is resiliently bounded: $\|\chi(t)\|_{\hat{P}} \leq \max\{0,\ b(t)\}$ for all $t \geq 0$, with

$$b(t) := p + h_+ e^{(r_+-\alpha_q)t} + h_- e^{(r_--\alpha_q)t}, \tag{11}$$

$$h_\pm = \frac{(\alpha_q - \alpha - r_\mp)\|\chi_0\|_{\hat{P}} + \gamma\|x_q^0\|_{P_q} + (r_\mp - \alpha_q)p}{\pm\sqrt{(\alpha_q - \alpha)^2 + 4\gamma\gamma_q}}$$

and $p = \frac{\gamma z_{max}^{P_q}}{\alpha\alpha_q - \gamma\gamma_q} > 0$.

Proof: The proof is given in Appendix III. ∎

Remark 3: Condition $\gamma\gamma_q < \alpha\alpha_q$ in Proposition 8 is necessary for the boundedness of $\chi(t)$, which in turn guarantees the admissibility of the control law $\hat{u}(t) = -K\chi(t)$.

In bound (20), the perturbation from nonresilient subsystem (3) is modeled by term $p > 0$. Because this perturbation is of constant magnitude, it cannot be overcome by the linear control $\hat{u}(t) = -K\chi(t)$ when χ is near 0. That is why Proposition 8 only guarantees the resilient boundedness of χ and not its resilient stabilizability.

Combining bounds (7) and (20), we can now derive a closed-form bound on x_q as we did in Proposition 7 when $\hat{B}$ was full rank.

Proposition 9: If pair $(\hat{A} + \hat{D}, \hat{B})$ is controllable, A_q is Hurwitz, $C_q\mathcal{W}_q \not\subseteq B_q\mathcal{U}_q$, $\gamma\gamma_q < \alpha\alpha_q$ and $m \leq \frac{\sqrt{\lambda_{min}^{\hat{P}}}}{\|K\|}$, then

$$\|x_q(t)\|_{P_q} \leq \begin{cases} \max\{0, e^{-\alpha_q t}\|x_q^0\|_{P_q} + \delta(t)\} & \text{if } \|\chi(t)\|_{\hat{P}} > 0 \\ \frac{z_{max}^{P_q}}{\alpha_q} + \left(\|x_q^0\|_{P_q} - \frac{z_{max}^{P_q}}{\alpha_q}\right) e^{-\alpha_q t} & \text{otherwise} \end{cases} \quad (12)$$

with

$$\delta(t) = \frac{\alpha z_{max}^{P_q}\left(1 - e^{-\alpha_q t}\right)}{\alpha\alpha_q - \gamma\gamma_q} + e^{-\alpha_q t} \sum_{* \in \{+,-\}} \frac{\gamma_q h_*}{r_*}\left(e^{r_* t} - 1\right).$$

Proof: The proof is extremely similar to that of Proposition 7 and can be found in the ArXiv version[1]. ∎

Using Propositions 8 and 9, we can now quantify the effect of the loss of control authority over nonresilient subsystem (3).

Without the full rank assumption on $\bar{B}$, we cannot resiliently stabilize the rest of the network (5), but we provide a guaranteed bound on its state χ. This constitutes our solution to Problem 2 for an underactuated network.

V. Numerical examples

We will now illustrate the theory established in the preceding sections on two academic examples, on an islanded microgrid [17]–[19], [30] and on the IEEE 39-bus system [31]. All the data and codes necessary to run the simulations in this section are available on GitHub[2].

A. Fully actuated 3-component network

We start by testing the results of Section IV-A on a simple network constituted of a nonresilient subsystem enduring a partial loss of control authority. This network of states χ_1, χ_2 and x_q follows dynamics

$$\dot{\chi}(t) = \begin{bmatrix} -1 & 0.3 \\ 0.3 & -1 \end{bmatrix}\chi(t) + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\hat{u}(t) + \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix}x_q(t), \quad (13)$$

$$\dot{x}_q(t) = -x_q(t) + u_q(t) + 2w_q(t) + \begin{bmatrix} 0.3 & 0.3 \end{bmatrix}\chi(t), \quad (14)$$

with $\chi(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $x_q(0) = 0$, $\hat{u}(t) = \begin{bmatrix} \hat{u}_1(t) \\ \hat{u}_1(t) \end{bmatrix} \in \hat{\mathcal{U}} = [-1, 1]^2$, $u_q(t) \in \mathcal{U}_q = [-1, 1]$ and $w_q(t) \in \mathcal{W}_q = [-1, 1]$. Following the notation of (3) and (5),

$$A_q = -1, \quad \hat{A} + \hat{D} = \begin{bmatrix} -1 & 0.3 \\ 0.3 & -1 \end{bmatrix}, \quad \text{and} \quad \hat{B} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}. \quad (15)$$

Matrices A_q and $\hat{A} + \hat{D}$ are both Hurwitz, and the control matrix $\hat{B}$ is full rank. Additionally, $C_q\mathcal{W}_q = [-2, 2] \not\subseteq B_q\mathcal{U}_q = [-1, 1]$. Thus, all the assumptions of Propositions 5, 6 and 7 are verified. To apply these results, we solve Lyapunov equations $A_q^\top P_q + P_q A_q = -Q_q$ and $(\hat{A}+\hat{D})^\top \hat{P} + \hat{P}(\hat{A}+\hat{D}) = -\hat{Q}$ with the function *lyap* on MATLAB:

$$Q_q = 1, \quad P_q = 0.5, \quad \hat{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } \hat{P} = \begin{bmatrix} 0.23 & 0.05 \\ 0.05 & 0.5 \end{bmatrix}.$$

Following Proposition 5, $\alpha_q = 1$ and $z_{max}^{P_q} = 1$ obtained for undesirable signal $w_q(t) = 1$ and control law $u_q(t) = -1$. From Proposition 6, $b_{min}^{\hat{P}} = 2$, $\alpha = 0.7$, $\gamma = 0.51$, and $\gamma_q = 0.48$. Then, the resilient stabilizability conditions of Theorem 3 are satisfied: $\gamma\gamma_q = 0.25 < \alpha\alpha_q = 0.7$ and $\gamma z_{max}^{P_q} = 0.5 < \alpha_q b_{min}^{\hat{P}} = 2$.

To verify that χ is indeed resiliently stabilizable in finite time by $\hat{B}\hat{u} = \frac{-\chi(t)}{\|\chi(t)\|_{\hat{P}}} b_{min}^{\hat{P}}$, we propagate $\chi(t)$ and $x_q(t)$. The finite-time resilient stabilization of state χ is illustrated on Fig. 1. Additionally, $\|\chi\|_{\hat{P}}$ satisfies the tight bound (8) at all times in this scenario.

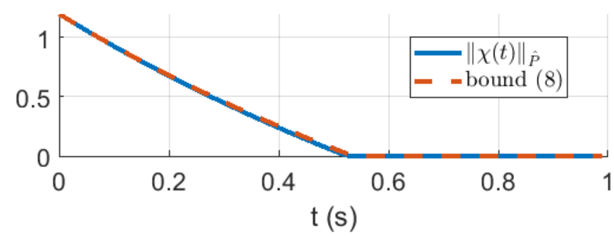

Fig. 1. Finite-time resilient stabilization of network state $\chi(t)$ of (13).

Fig. 2 shows that malfunctioning state x_q cannot be maintained at the origin but that its norm is bounded by both bounds (7) and (10). Since $\alpha_q > 0$ and $r_\pm < \alpha_q$, both bounds (7) and (10) converge, and hence so does x_q, which is then resiliently bounded. As discussed in Remark 2, when $\chi(t)$ reaches 0, bound (10) operates a discontinuous switch visible on Fig. 2 around $t = 0.5\,s$.

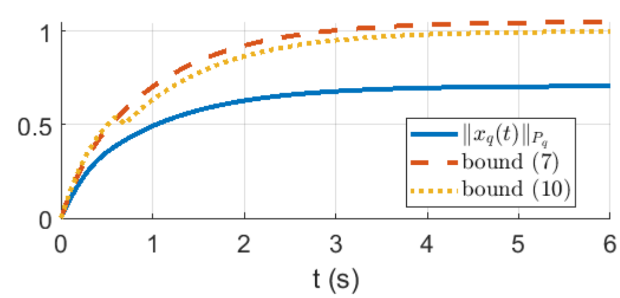

Fig. 2. Resiliently bounded malfunctioning state $x_q(t)$ of (14).

[2]https://github.com/Jean-BaptisteBouvier/Network-Resilience

B. Underactuated 3-component network

To validate the results of Section IV-B, we need $\hat{B}$ not to be full row rank anymore, but the pair $(\hat{A}+\hat{D}, \hat{B})$ must remain controllable. Then, we remove the second column of $\hat{B}$ so that (13) becomes

$$\dot{\chi}(t) = \begin{bmatrix} -1 & 0.3 \\ 0.3 & -1 \end{bmatrix} \chi(t) + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \hat{u}(t) + \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix} x_q(t), \qquad (16)$$

and (14) is left unchanged. The MATLAB functions *lqr* and *lyap* choose the following gain matrix K and positive definite matrices $\hat{P}$ and $\hat{Q}$:

$$K = \begin{bmatrix} 0.64 & 0.15 \end{bmatrix}, \quad \hat{P} = \begin{bmatrix} 0.22 & 0.04 \\ 0.04 & 0.5 \end{bmatrix}, \text{ and } \hat{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then, $\gamma\gamma_q = 0.24 < \alpha\alpha_q = 0.98$ and $\hat{u}(t) := -K\chi(t) \in [-1, 1]$. Thus, the linear feedback of Proposition 8 is admissible and its bound (20) holds as shown on Fig. 3.

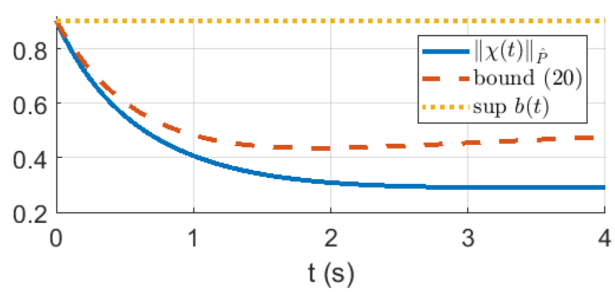

Fig. 3. Resiliently bounded network state $\chi(t)$ of (16).

On the contrary to bound (10) on Fig. 2, bound (12) on Fig. 4 does not switch. Indeed, χ cannot be brought to 0 by the linear control $\hat{u}$, as explained after Proposition 8. As illustrated on Fig. 4, bound (7) is tighter than bound (12). The reason for this difference in conservatism is that (7) uses directly the value of χ, while (12) replaces χ by its bound (20). As in Fig. 2, x_q is resiliently bounded since $r_\pm < \alpha_q$ guarantees the convergence of bound (12).

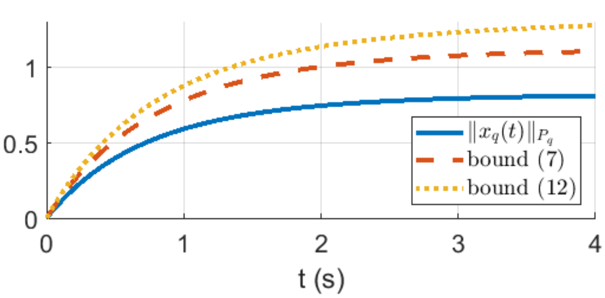

Fig. 4. Resiliently bounded malfunctioning state $x_q(t)$.

To verify the admissibility of the linear control law $\hat{u}(t) = -K\chi(t)$, we cannot use the sufficient condition of Proposition 8 as $\sup b(t) = 0.9 > \frac{\sqrt{\lambda_{min}^{\hat{P}}}}{\|K\|} = 0.71$. However, we can see on Fig. 5 that $\|K\chi(t)\| \leq 1$ for all $t \geq 0$ and thus $\hat{u}$ is in fact admissible. Note that $\hat{u}(t)$ does not converge to 0 since it needs to constantly counteract the destabilizing impact of $x_q(t)$ on $\chi(t)$.

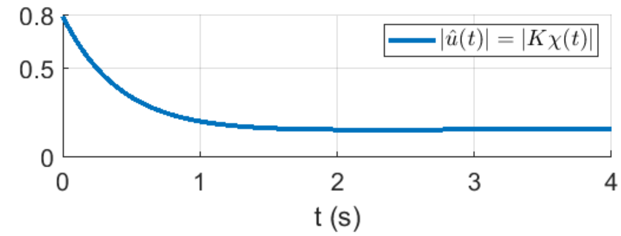

Fig. 5. Admissible linear feedback $\hat{u}(t) = -K\chi(t)$.

C. Microgrid test system

We will now investigate the resilient stabilizability of the islanded microgrid illustrated on Fig. 6 and studied in numerous power system works such as [17]–[19], [30].

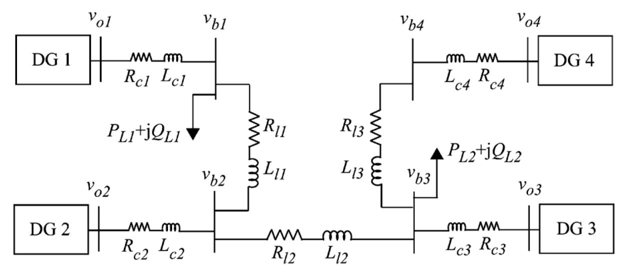

Fig. 6. Single-line diagram of the microgrid test system from [17].

Distributed generator 1 (DG1) is connected to the leader node DG0 with pinning gain 1 and all the DGs are connected following the communication digraph of Fig. 7.

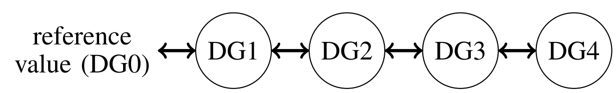

Fig. 7. Topology of the communication digraph.

We follow works [17]–[19] and employ their input-output feedback linearization of the DG dynamics. Since all DGs aim at synchronizing their voltage to the reference $v_0 = v_{ref}$, we consider as states the voltage difference between neighbors: $x_i := \begin{bmatrix} v_i - v_{i-1}, \dot{v}_i - \dot{v}_{i-1} \end{bmatrix}^\top$ for $i \in [\![1, 4]\!]$. Then, the objective is to stabilize all the x_i to the origin. After a loss of control authority in DG4, we instead aim at bounding the voltages so that they do not diverge too far from the reference. The linearized microgrid is underactuated but controllable with $\gamma\gamma_q = 0.0399 < \alpha\alpha_q = 0.0401$, so that we can apply Propositions 8 and 9.

As seen on Fig. 8, bounds (7) and (12) are initially tight and only diverge slowly from $\|x_q(t)\|_{P_q}$.

Since DG4 is not resilient, v_4 does not converge to the reference v_0, but is nonetheless bounded as shown on Fig. 9. In turn, DG4 disrupts its neighbor DG3, whose voltage v_3 cannot reach v_0 either. However, v_3 is maintained much closer to v_0 than v_4 thanks to the resilient controller of Proposition 8. This resilient controller also allows DG1 and DG2 to remain completely oblivious of the loss of control of DG4.

D. Resilient stabilizability of the IEEE 39-bus system

We will now investigate the resilient stabilizability of the IEEE 39-bus system [31] illustrated on Fig. 10. This system is

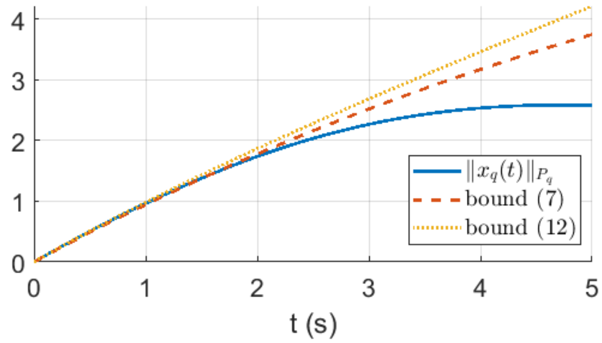

Fig. 8. Bounded malfunctioning state $\boldsymbol{x_q} = [\boldsymbol{v_4} - \boldsymbol{v_3}, \dot{\boldsymbol{v}}_4 - \dot{\boldsymbol{v}}_3]^\top$ with tight bounds (7) and (12).

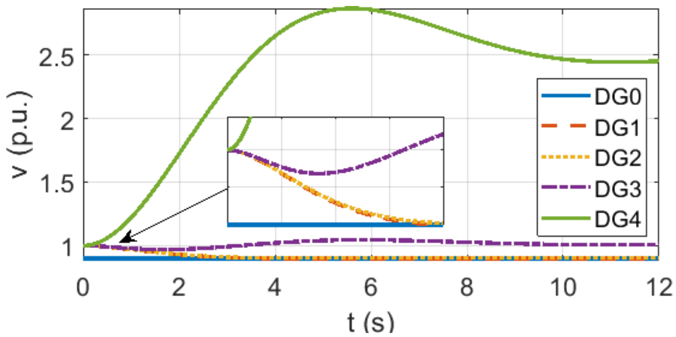

Fig. 9. Magnitude of the output voltages of the reference and the four nodes of the microgrid.

comprised of 29 load buses and 10 generator buses. The state of the load buses is described solely by their phase angles $\{\delta_i\}_{i \in [\![1,29]\!]}$, while the state of the generators is composed of phase angles $\delta_{i \in [\![30,39]\!]}$ and frequencies $\omega_{i \in [\![30,39]\!]}$, which leads to a total of 49 states. Only the generator buses possess a control input. Following [20], the power network equations can be linearized around their nominal operating point after adjustment for the reference bus, chosen to be the first generator, i.e., bus 30. The state vector is then

$$x = \big(\{\delta_i - \delta_{30}\}_{i \in [\![1,29]\!] \cup [\![31,39]\!]}, \{\omega_i\}_{i \in [\![30,39]\!]}\big) \in \mathbb{R}^{48}.$$

After a cyber-attack, the network controller loses control authority over generator bus 39, i.e., $x_q = [\delta_{39}\ \omega_{39}]^\top$ and $w_q = u_{39}$. Following [20], the malfunctioning dynamics are

$$\begin{bmatrix} \dot{\delta}_{39}(t) \\ \dot{\omega}_{39}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -18.6 & -11.2 \end{bmatrix} \begin{bmatrix} \delta_{39}(t) \\ \omega_{39}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.22 \end{bmatrix} w_q(t) + D_{q,\text{-}}\chi(t).$$

We choose initial states $\chi(0) = \mathbf{1}_{46}$, $\delta_{39}(0) = 0$ rad and $\omega_{39}(0) = 0$ Hz. Since A_q is Hurwitz and $B_q = 0$, the assumptions of Proposition 5 are satisfied. Additionally, pair $(\hat{A} + \hat{D}, \hat{B})$ is controllable so we can find a stabilizing gain matrix K for the network dynamics. However, we cannot apply Proposition 8 because the resilient stability condition $\gamma\gamma_q < \alpha\alpha_q$ is not satisfied. Indeed, $\gamma\gamma_q = 6.3 \times 10^4$, while $\alpha\alpha_q = 5.7 \times 10^{-3}$. This magnitude difference leads to the exponential divergence of bounds (12) and (20), as seen on Fig. 11.

Note that bound (7) is much tighter than (12) because bound (7) uses $\int_0^t e^{-\alpha_q(t-\tau)} \|D_{q,\text{-}}\chi(\tau)\|_{P_q} d\tau$, whereas (7) bounds this integral with exponentially diverging (20). In fact, bound (7) remains a reasonable bound for malfunctioning state x_q over a much longer time horizon as illustrated on Fig. 12.

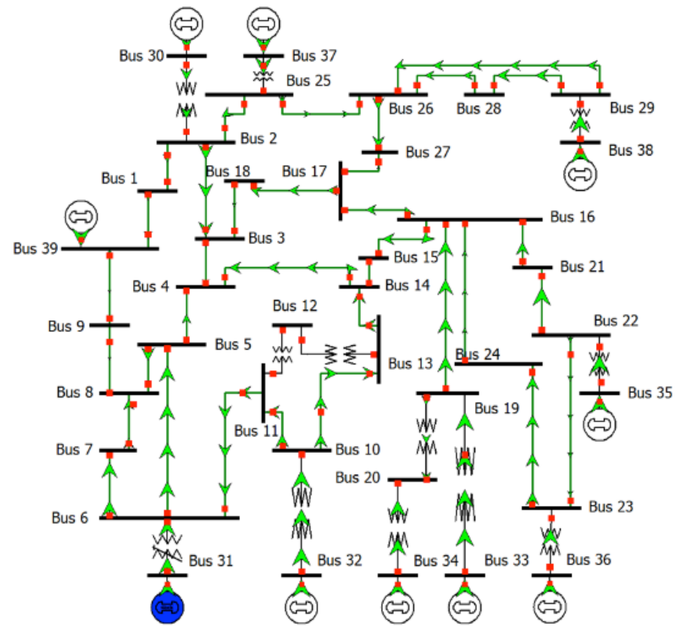

Fig. 10. Illustration of the IEEE 39-bus system [31] obtained from `https://icseg.iti.illinois.edu/ieee-39-bus-system/`.

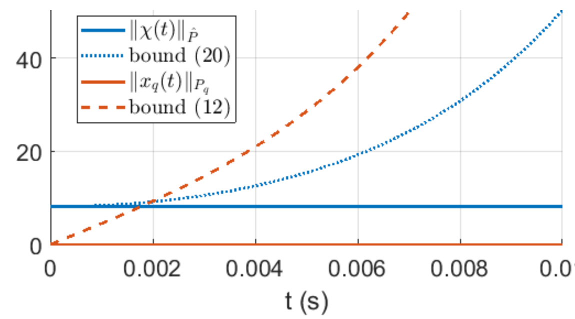

Fig. 11. Simulation of the network state $\boldsymbol{\chi}$ with its bound (20) and malfunctioning state $\boldsymbol{x_q}$ with its bound (12). Both bounds are exponentially diverging because $\boldsymbol{\gamma\gamma_q} \gg \boldsymbol{\alpha\alpha_q}$, which contradicts the stability condition of Proposition 8.

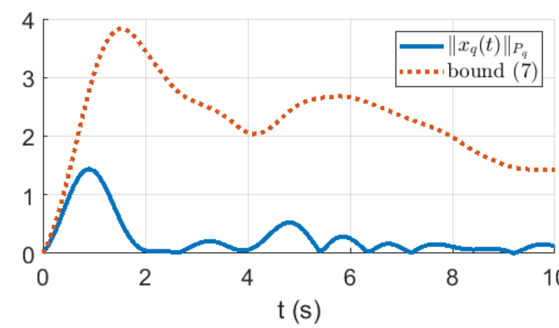

Fig. 12. Simulation of malfunctioning state $\boldsymbol{x_q}$ of the IEEE 39-bus system with bound (7).

As before, sufficient condition $\sup\limits_{t \geq 0} b(t) \leq \frac{\sqrt{\lambda_{min}^{\hat{P}}}}{\|K\|}$ of Proposition 8 cannot tell whether linear feedback $\hat{u}$ is admissible. However, the choice of K ensures admissibility

$\max_{i,t}|K\chi_i(t)| \leq 1$ as shown on Fig. 13.

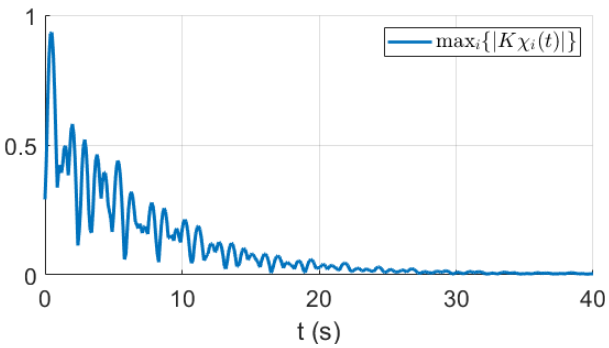

Fig. 13. Maximal component of the linear feedback $\hat{u}(t) = -K\chi(t)$.

Let us delve a bit deeper into the exponential divergence of bound (20). As mentioned previously, bound (20) is not tight because $\gamma\gamma_q = 6.3 \times 10^4$ is orders of magnitude larger than $\alpha\alpha_q = 5.7 \times 10^{-3}$, whereas the stability condition of Proposition 8 calls for $\gamma\gamma_q < \alpha\alpha_q$. As discussed after Theorem 3, this condition carries the intuition that the perturbations arising from the coupling between x_q and χ should be weaker than their respective stability. Despite having $\gamma\gamma_q \gg \alpha\alpha_q$, the coupling does not destabilize states x_q and χ, which are both bounded, as shown on Fig. 14.

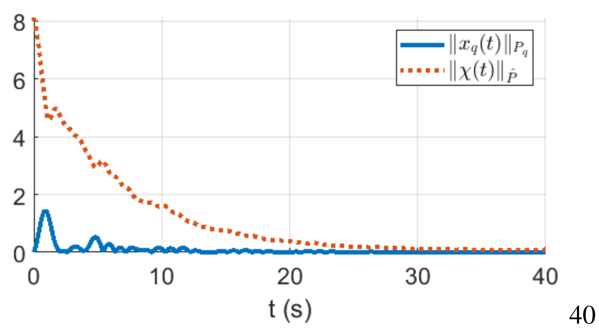

Fig. 14. Simulation of network state χ and malfunctioning state x_q of the IEEE 39-bus system.

Since the coupling does not destabilize states χ and x_q, the violation of stability condition $\gamma\gamma_q < \alpha\alpha_q$ is in fact due to the failure of parameters γ and γ_q to characterize the coupling between states χ and x_q. As shown on Fig. 10 each bus is only connected to a small number of other buses. Then, matrix $\hat{D}$ is almost entirely composed of zeros except for a handful of terms per row. Because of this strong coupling with very few nodes, constants γ and γ_q are very large. However, the sparsity of matrix $\hat{D}$ results in weak coupling of states χ and x_q, rendering γ and γ_q overly conservative. This intuition was illustrated on the more densely connected microgrid of Section V-C, where the bounds were much tighter. To study sparsely connected networks like the IEEE 39-bus system, we have the intuition that choosing a different norm reflecting the sparsity of matrix $\hat{D}$ would lower the values of γ and γ_q. Doing so would significantly and non-trivially alter all the proofs of Section IV.

VI. Conclusion and Future Work

This paper investigated the resilient stabilizability of linear networks enduring a loss of control. We first saw that the overall stabilizability of networks composed exclusively of resilient subsystems depends only on their interconnection. Then, we focused on networks losing control authority over a nonresilient subsystem. In this scenario, we showed that under some conditions, the state of underactuated networks can remain bounded and the state of fully actuated networks can be stabilized. We were able to quantify the maximal magnitude of undesirable inputs that can be applied to a nonresilient subsystem without destabilizing the rest of the network.

We are considering several avenues of future work. First, building on the nonlinear resilience theory of [22], we would like to extend our approach to nonlinear networks. Doing so would allow us to study the true nonlinear dynamics of power systems, including the IEEE 39-bus system. Second, following the discussion at the end of Section V-D, we want to extend this theory to different matrix norms to provide tighter bounds for sparse coupling matrices. The last avenue of future work would be to relax the assumption of real-time knowledge of the undesirable inputs by the controller. Doing so would allow to account for actuation delays and can possibly be accomplished following the techniques introduced in [22].

Appendix I
Proof of Proposition 6

Proof: Since $\hat{A} + \hat{D}$ is Hurwitz, there exist a symmetric $\hat{P} \succ 0$ and $\hat{Q} \succ 0$ such that $(\hat{A} + \hat{D})^\top \hat{P} + \hat{P}(\hat{A} + \hat{D}) = -\hat{Q}$ according to Lyapunov theory [28]. Following the same steps as in the proof of Proposition 5 with $\chi^\top \hat{P} \chi = \|\chi\|^2_{\hat{P}}$ and χ following the dynamics (5), we first obtain

$$\frac{d}{dt}\|\chi(t)\|^2_{\hat{P}} = -\chi(t)^\top \hat{Q}\chi(t) + 2\chi(t)^\top \hat{P}\big(\hat{B}\hat{u}(t) + D_{\text{-},q}x_q(t)\big).$$

Because $\hat{B}$ is full rank, for all $\chi(t) \neq 0$ there exist $\hat{u}(t) \in \hat{\mathcal{U}}$ such that $\hat{B}\hat{u}(t) = -\frac{\chi(t)}{\|\chi(t)\|_{\hat{P}}} b^{\hat{P}}_{min}$, as shown in the proof of Proposition 5 of [10]. Then,

$$\chi(t)^\top \hat{P}\hat{B}\hat{u}(t) = -\frac{\chi(t)^\top \hat{P}\chi(t)}{\|\chi(t)\|_{\hat{P}}} b^{\hat{P}}_{min} = -\|\chi(t)\|_{\hat{P}} b^{\hat{P}}_{min}.$$

Since $\|\cdot\|_{\hat{P}}$ is a norm, it verifies the Cauchy-Schwarz inequality [27] $\chi^\top \hat{P} D_{\text{-},q}x_q \leq \|\chi\|_{\hat{P}} \|D_{\text{-},q}x_q\|_{\hat{P}}$. Then,

$$\frac{d}{dt}\|\chi(t)\|^2_{\hat{P}} \leq -\chi(t)^\top \hat{Q}\chi(t) - 2\|\chi(t)\|_{\hat{P}} b^{\hat{P}}_{min} + 2\|\chi(t)\|_{\hat{P}} \|D_{\text{-},q}x_q(t)\|_{\hat{P}}.$$

As in Proposition 5, $\hat{P} \succ 0$ and $\hat{Q} \succ 0$ yield $-\chi^\top \hat{Q}\chi \leq -\frac{\lambda^{\hat{Q}}_{min}}{\lambda^{\hat{P}}_{max}}\|\chi\|^2_{\hat{P}}$. Since $\hat{P}^\top = \hat{P} \succ 0$ and $\hat{P}_q \succ 0$, applying the Rayleigh quotient inequality [27] yields $\|D_{\text{-},q}x_q\|_{\hat{P}} \leq \gamma\|x_q\|_{P_q}$. We now combine these inequalities into

$$\frac{d}{dt}\|\chi\|^2_{\hat{P}} \leq -\frac{\lambda^{\hat{Q}}_{min}}{\lambda^{\hat{P}}_{max}}\|\chi\|^2_{\hat{P}} + 2\|\chi\|_{\hat{P}}\big(\gamma\|x_q\|_{P_q} - b^{\hat{P}}_{min}\big).$$

Let $y(t) := \|\chi(t)\|_{\hat{P}}$ and $y_q(t) := \|x_q(t)\|_{P_q}$. Then, following Proposition 5, we include bound (7), which yields

$$\frac{d}{dt}y(t)^2 \leq -\frac{\lambda_{min}^{\hat{Q}}}{\lambda_{max}^{\hat{P}}}y(t)^2 - 2y(t)b_{min}^{\hat{P}} + 2y(t)\gamma e^{-\alpha_q t}\left(y_q(0) + \int_0^t e^{\alpha_q \tau}\beta_q(\tau)d\tau\right).$$

Since $P_q^\top = P_q \succ 0$ and $\hat{P} \succ 0$, applying the Rayleigh quotient inequality [27] yields $\|D_{q,\text{-}}\chi\|_{P_q} \leq \gamma_q\|\chi\|_{\hat{P}}$, which can be used in β_q defined in Proposition 5 as

$$\beta_q(\tau) = z_{max}^{P_q} + \|D_{q,\text{-}}\chi(\tau)\|_{P_q} \leq z_{max}^{P_q} + \gamma_q y(\tau). \tag{17}$$

Noticing $\frac{d}{dt}y(t)^2 = 2y(t)\dot{y}(t)$, for $y(t) > 0$ we divide both sides of the inequality preceding (17) by $2y(t)$, which yields

$$\dot{y}(t) \leq -\alpha y(t) - b_{min}^{\hat{P}} + \gamma y_q(0)e^{-\alpha_q t} + \gamma e^{-\alpha_q t}\int_0^t e^{\alpha_q \tau}\left(z_{max}^{P_q} + \gamma_q y(\tau)\right)d\tau.$$

We calculate the following trivial integral

$$e^{-\alpha_q t}\int_0^t e^{\alpha_q \tau}\,d\tau = e^{-\alpha_q t}\frac{e^{\alpha_q t}-1}{\alpha_q} = \frac{1-e^{-\alpha_q t}}{\alpha_q},$$

so that the differential inequality becomes

$$\dot{y}(t) \leq -\alpha y(t) + \frac{\gamma z_{max}^{P_q}}{\alpha_q} - b_{min}^{\hat{P}} + \gamma\left(y_q(0) - \frac{z_{max}^{P_q}}{\alpha_q}\right)e^{-\alpha_q t} + \gamma\gamma_q e^{-\alpha_q t}\int_0^t e^{\alpha_q \tau}y(\tau)\,d\tau.$$

Now multiply both sides by $e^{\alpha_q t} > 0$ and define $v(t) := e^{\alpha_q t}y(t)$. Then, $\dot{v}(t) = \alpha_q v(t) + e^{\alpha_q t}\dot{y}(t)$, which leads to $\dot{v}(t) \leq f\big(t, v(t)\big)$ with function

$$f\big(t,s(t)\big) := (\alpha_q - \alpha)s(t) + \left(\frac{\gamma z_{max}^{P_q}}{\alpha_q} - b_{min}^{\hat{P}}\right)e^{\alpha_q t} + \gamma\left(y_q(0) - \frac{z_{max}^{P_q}}{\alpha_q}\right) + \gamma\gamma_q\int_0^t s(\tau)\,d\tau.$$

Now, we search for a solution to the differential equation $\dot{s}(t) = f\big(t, s(t)\big)$. Differentiating this equation yields

$$\ddot{s}(t) + (\alpha - \alpha_q)\dot{s}(t) - \gamma\gamma_q s(t) - \left(\gamma z_{max}^{P_q} - \alpha_q b_{min}^{\hat{P}}\right)e^{\alpha_q t} = 0. \tag{18}$$

We distinguish two cases when solving (18). If $\alpha\alpha_q \neq \gamma\gamma_q$, $s(t) = pe^{\alpha_q t} + h_+e^{r_+ t} + h_- e^{r_- t}$, with $p = \frac{\gamma z_{max}^{P_q} - \alpha_q b_{min}^{\hat{P}}}{\alpha\alpha_q - \gamma\gamma_q}$, $h_\pm \in \mathbb{R}$ two constants and

$$r_\pm = \frac{1}{2}\left(\alpha_q - \alpha \pm \sqrt{(\alpha-\alpha_q)^2 + 4\gamma\gamma_q}\right). \tag{19}$$

We apply the Comparison Lemma of [29] stating that if $\dot{s}(t) = f\big(t, s(t)\big)$, f is continuous in t and locally Lipschitz in s and $s(0) = v(0)$, then $\dot{v}(t) \leq f\big(t, v(t)\big)$ implies $v(t) \leq s(t)$ for all $t \geq 0$. Using $\|\chi(t)\|_{\hat{P}} = y(t) = e^{-\alpha_q t}v(t) \leq e^{-\alpha_q t}s(t)$, we obtain (8). To determine $h_\pm$, we use the initial conditions $s(0) = v(0) = y(0)$ and $\dot{s}(0) = f\big(0, s(0)\big)$, which yield

$$h_\pm = \frac{(\alpha_q - \alpha - r_\mp)\|\chi_0\|_{\hat{P}} + \gamma\|x_q^0\|_{P_q} - b_{min}^{\hat{P}} + (r_\mp - \alpha_q)p}{\pm\sqrt{(\alpha-\alpha_q)^2 + 4\gamma\gamma_q}}.$$

In the case $\alpha\alpha_q = \gamma\gamma_q$, the solution of (18) is

$$s(t) = pte^{\alpha_q t} + h_+e^{\alpha_q t} + h_- e^{-\alpha t}$$

with $p = \frac{\gamma z_{max}^{P_q} - \alpha_q b_{min}^{\hat{P}}}{\alpha + \alpha_q}$, and

$$h_\pm = \frac{\frac{1}{2}\big(-\alpha_q - \alpha \pm 3(\alpha - \alpha_q)\big)\|\chi_0\|_{\hat{P}} \mp \gamma\|x_q^0\|_{P_q} \pm b_{min}^{\hat{P}} \pm p}{\alpha_q + \alpha}.$$

Applying the Comparison Lemma of [29] as above, we obtain $\|\chi(t)\|_{\hat{P}} = y(t) = e^{-\alpha_q t}v(t) \leq e^{-\alpha_q t}s(t)$ yielding (9). ■

APPENDIX II
PROOF OF PROPOSITION 7

Proof: We start with the case $\|\chi(t)\|_{\hat{P}} > 0$. Then, bound (8) combined with (17) yields

$$\int_0^t e^{\alpha_q \tau}\beta_q(\tau)d\tau \leq \int_0^t e^{\alpha_q \tau}\Big(z_{max}^{P_q} + \gamma_q p + \gamma_q\sum_{*\in\{+,-\}} h_* e^{(r_* - \alpha_q)\tau}\Big)d\tau = \frac{e^{\alpha_q t}-1}{\alpha_q}\big(z_{max}^{P_q} + \gamma_q p\big) + \sum_{*\in\{+,-\}}\big(e^{r_* t} - 1\big)\frac{\gamma_q h_*}{r_*}.$$

Using $p = \frac{\gamma z_{max}^{P_q} - \alpha_q b_{min}^{\hat{P}}}{\alpha\alpha_q - \gamma\gamma_q}$ yields $\frac{z_{max}^{P_q} + \gamma_q p}{\alpha_q} = \frac{\alpha z_{max}^{P_q} - \gamma_q b_{min}^{\hat{P}}}{\alpha\alpha_q - \gamma\gamma_q}$. Then, plugging the integral calculated above in (7), we obtain

$$\|x_q(t)\|_{P_q} \leq e^{-\alpha_q t}\Bigg(\|x_q^0\|_{P_q} + \frac{\alpha z_{max}^{P_q} - \gamma_q b_{min}^{\hat{P}}}{\alpha\alpha_q - \gamma\gamma_q}\big(e^{\alpha_q t} - 1\big) + \frac{\gamma_q h_+}{r_+}\big(e^{r_+ t} - 1\big) + \frac{\gamma_q h_-}{r_-}\big(e^{r_- t} - 1\big)\Bigg),$$

which yields (10). When $\|\chi(t)\|_{\hat{P}} = 0$, β_q simplifies to $z_{max}^{P_q}$ and yields

$$\|x_q(t)\|_{P_q} \leq e^{-\alpha_q t}\left(\|x_q^0\|_{P_q} + \int_0^t e^{\alpha_q \tau}z_{max}^{P_q}\,d\tau\right) = \frac{z_{max}^{P_q}}{\alpha_q} + \left(\|x_q^0\|_{P_q} - \frac{z_{max}^{P_q}}{\alpha_q}\right)e^{-\alpha_q t}.$$

■

APPENDIX III
PROOF OF PROPOSITION 8

Proof: We will start by obtaining bounds on $\chi(t)$ with control law $\hat{u}(t) = -K\chi(t)$ and then, we will verify under which conditions is this $\hat{u}$ admissible. We follow the same steps as in the proof of Proposition 6 with $\chi^\top\hat{P}\chi = \|\chi\|_{\hat{P}}^2$ and χ following the dynamics (5). Applying control law $\hat{u}(t) = -K\chi(t)$ to subsystem (5) leads to

$$\dot{\chi}(t) = \big(\hat{A} - \hat{B}K + \hat{D}\big)\chi(t) + D_{\text{-},q}x_q(t).$$

Lyapunov equation $(\hat{A} - \hat{B}K + \hat{D})^\top\hat{P} + \hat{P}(\hat{A} - \hat{B}K + \hat{D}) = -\hat{Q}$ now yields

$$\frac{d}{dt}\|\chi(t)\|_{\hat{P}}^2 = -\chi(t)^\top\hat{Q}\chi(t) + 2\chi(t)^\top\hat{P}D_{\text{-},q}x_q(t).$$

We then proceed as in Proposition 6, except that we do not have the term $b_{min}^{\hat{P}}$ anymore, which leads to

$$\|\chi(t)\|_{\hat{P}} \leq \max\left\{0,\ p + h_+e^{(r_+ - \alpha_q)t} + h_- e^{(r_- - \alpha_q)t}\right\} \tag{20}$$

as long as $\hat{u}(t)$ is admissible. Admissibility occurs while $\hat{u}(t) = -K\chi(t) \in \hat{\mathcal{U}} = [-1, 1]^{m_\Sigma - m_q}$. By assumption,

$$\|\hat{u}(t)\| \leq \|K\|\|\chi(t)\| \leq \|K\| \frac{\|\chi(t)\|_{\hat{P}}}{\sqrt{\lambda^{\hat{P}}_{min}}} \leq \sup_{t \geq 0} b(t) \frac{\|K\|}{\sqrt{\lambda^{\hat{P}}_{min}}} \leq 1.$$

Therefore $\hat{u}(t)$ is admissible. Since $\gamma\gamma_q < \alpha\alpha_q$, we have $r_\pm < \alpha_q$, so the two exponentials of (20) are bounded, i.e., χ is resiliently bounded. ■

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